The disorder-interaction problem

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Overview

- Introduction: Noninteracting systems
  - Theoretical concepts
  - Wigner-Dyson spectral statistics
- Correlated disordered systems
  - Bosons in one dimension
    - Interference in interacting clean 1D Bose gas
    - Disordered strongly interacting Bose gas: Bose-Fermi mapping to noninteracting fermions (Anderson insulator)
  - Replica Field Theory (sigma models)
    - Local density of states in disordered multichannel wires

Disorder in noninteracting systems

- Quantum coherent systems
- Some manifestations of phase coherence in mesoscopic structures:
  - Universal conductance fluctuations (UCF), absence of self-averaging
  - Weak localization: Enhanced return probability
  - Spectral fluctuations, level statistics
- Why can interactions often be neglected?

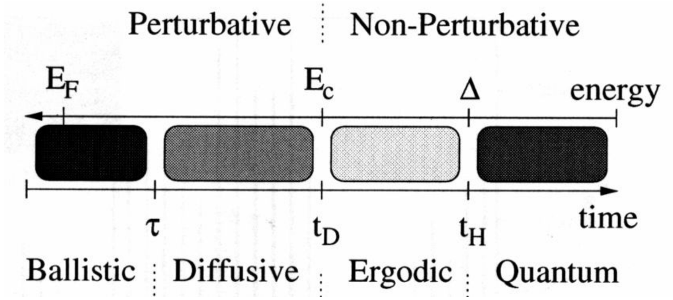
Fermi liquid theory

- In normal metals, interactions lead to formation of Landau quasiparticles (Fermi liquid)
  - Weakly interacting Fermions, stable at low energies
  - Quasiparticle relaxation rate due to interactions $\Gamma \propto E^2$
  - In disordered systems more dangerous: $\Gamma \propto E^{d/2}$
- Standard picture in mesoscopics, usually neglect of interactions protected by Fermi liquid principle
- But: Breakdown of Fermi liquid possible
  - New physical effects
  - New methods required

Methods for noninteracting systems

- Semiclassical techniques
  - Restricted to essentially clean (chaotic) systems
- Diagrammatic perturbation theory
  - Breaks down in nonperturbative regime
- Random matrix theory
  - Wigner-Dyson ensembles and generalization *(d=0)*
  - 1D multimode wires: Transfer matrix ensembles (DMPK)
- Field theories (nonlinear sigma model)
  - Supersymmetric formulation (Efetov)
  - Replica/Keldysh field theory (Wegner, Finkel'stein)
- Special techniques
  - Berezinskii diagram technique in 1D
  - Fisher RG scheme for disordered spin chains

Time and energy scales



- Ballistic particle motion up to mean free time $t < \tau$
- Diffusion for *E* > Thouless energy $E_c = \hbar D / L^2$

  $$\tau < t < t_D = L^2 / D, D = u^2 \tau / d$$

- Ergodic regime $t_D < t < t_H = \hbar / \Delta$

  $$\Delta = 1 / \nu L^d$$

  - Wavefunctions probe the whole system
  - Universal regime, only governed by symmetries
- Resolution of single particle levels at lowest energy scales: Quantum regime $t > t_H = \hbar / \Delta$
- Nonperturbative regime not captured by most methods, easy to miss…

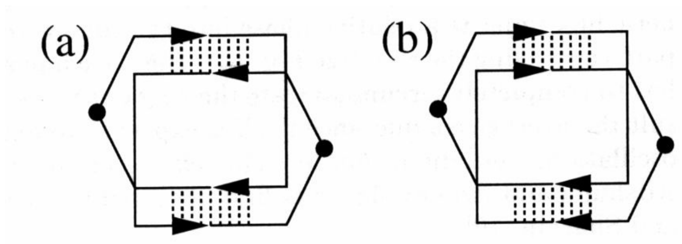
Energy level repulsion & universality

- Main interest in mesoscopics: Transport quantities (conductance, shot noise)
- Phase coherence also causes characteristic fluctuations in spectral properties
- Universal & nonperturbative physics
  - only controlled by symmetries and number of accessible states
- Two-point correlations of DoS fluctuations

$$R_2(\Omega) = \Delta^2 \langle \rho(E+\Omega/2)\rho(E-\Omega/2)\rangle_{qm,dis} - 1$$

$$\rho(E) = Tr\,\delta(E-H)$$

Spectral correlations



- Diagrammatics: Diffuson (a) and possibly Cooperon (b) show unphysical divergence from zero mode

$$R_{2,(a)}(\Omega) = \frac{1}{2\pi^2} \mathrm{Re} \sum_q \frac{\Delta^2}{\left(-i\Omega + Dq^2\right)^2}$$

- Exact result (here: broken time-reversal invariance) covers nonperturbative regime, no artificial divergence

Oscillatory Wigner-Dyson correlations

$$R_2(\Omega) = -\frac{\sin^2(\pi\Omega/\Delta)}{(\pi\Omega/\Delta)^2}$$

- Experimental observation in cold atom systems?

Concepts for interacting systems

- Many of these methods not applicable anymore…
  - Supersymmetry
  - DMPK approach, Berezinskii method
  - Semiclassics, standard RMT models

- … or only perturbative results: Diagrammatic theory

  Disorder enhanced interaction effects, zero-bias anomalies (ZBA) *Altshuler & Aronov, 1980*
- Approaches that (can) work:

  - Luttinger liquid theory (1D), exactly solvable in clean case
  - Interacting nonlinear σ model: Replica/Keldysh field theory
  - 1D dirty bosons with strong interactions: Bose-Fermi mapping to Anderson localization of free fermions

Luttinger liquid: 1D gapless systems

Luttinger, JMP 1963
Haldane, JPC 1981

- Abelian Bosonization: Field $\Phi(x,\tau)\in S^1$
  - Field describes charge or spin density
  - Free Gaussian field theory, interactions are nonperturbatively included in *g* and velocity *u*

$$S_{LL}[\Phi]=\frac{1}{2\pi g}\int dxd\tau\left(\frac{1}{u}(\partial_\tau\Phi)^2+u(\partial_x\Phi)^2\right)$$

- Clean case: Exactly solvable
- Disorder strongly relevant, localization
- Multichannel generalization possible

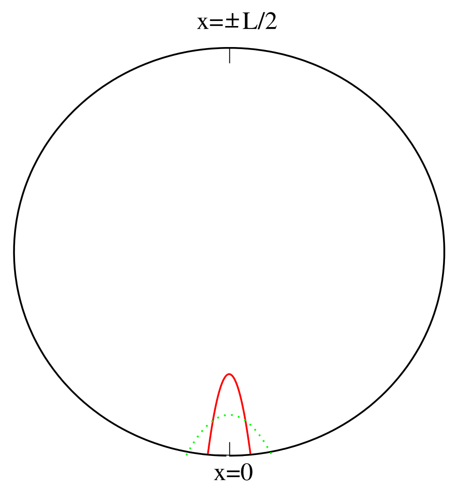
Luttinger liquid phenomena

- No Landau quasiparticles, but Laughlin-type quasiparticles (solitons of field theory)
  - Anyon statistics, fractional charge
  - Should be easier to probe in cold atom systems (no leads attached!)
- Spin-charge separation
  - Proposals for cold atoms exist *Recati et al., PRL 2003*
- Applies to Bosons and Fermions
  - Interference of interacting 1D Bose atom waves

Bosons in 1D traps: Interference

Chen & Egger, PRA 2003

- Mach-Zehnder-type interferometer for Bose atom wavepackets
- Axial trap potential switched off at *t=0*, nonequilibrium initial state
- Expansion, then interference at opposite side
- Interference signal
  - Dependence on interactions?
  - Dependence on temperature?



Theoretical description

1D Bose gas on a ring with time-dependent axial potential *V(x,t)*

- Exact Lieb-Liniger solution only without potential
- Low energy limit & gradient expansion (LDA) yields generalized Luttinger liquid
- Quadratic in density & phase fluctuations

$$\Pi(x,t) = \rho(x,t) - \rho_0(x,t)$$

$$\Phi(x,t) = \phi(x,t) - \phi_0(x,t)$$

around solution of GP equation

$$\psi_0(x,t) = \sqrt{\rho_0}\, e^{i\phi_0}$$

Hamiltonian

- Luttinger type Hamiltonian:

$$H(t) = \int dx \left[\frac{\hbar^2 \rho_0}{2m_0} (\partial_x \Phi)^2 + \frac{\hbar^2}{m_0} \partial_x \phi_0 \;\; \Pi \partial_x \Phi + \frac{\partial F(\rho_0)}{\partial \rho_0} \frac{\Pi^2}{2} \right]$$

$$F(\rho_0) = \begin{cases} g_B \rho_0, & TF : g_B \to 0 \\ \pi^2 \hbar^2 \rho_0^2 / 2m_0, & TG : g_B \to \infty \end{cases}$$

- Quadratic Hamiltonian, can be diagonalized for any time-dependent potential
- Time-dependent & non-uniform

Interference signal

Consider $$V(x,t)=\begin{cases}\dfrac{m_0\omega_x^2 x^2}{2}, & t<0\\ 0, & t>0\end{cases}$$

and self-similar limits:

Thomas-Fermi (TF) or

Tonks-Girardeau (TG)

$$\rho_0(x,t)=\frac{\rho_0\left(x/b(t),0\right)}{b(t)}$$

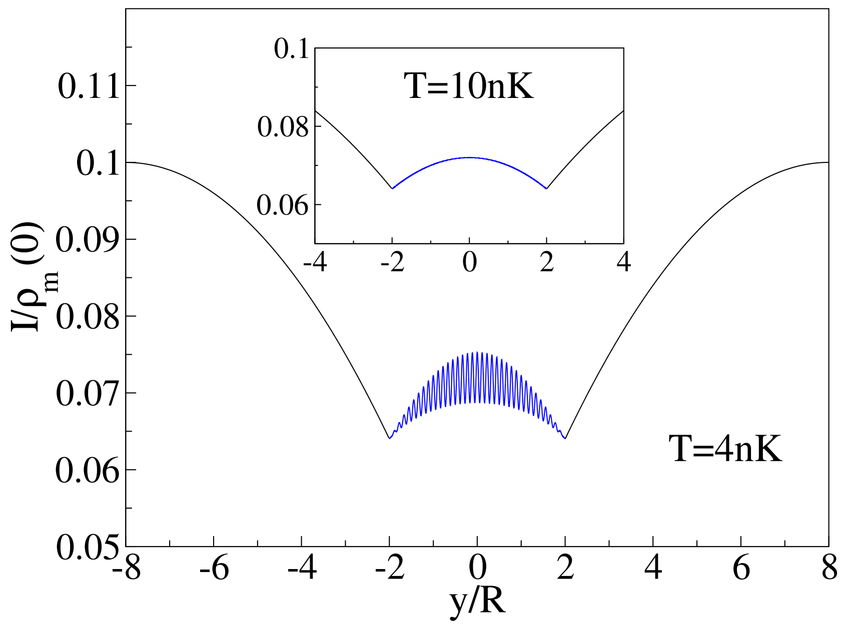
with known scale function $b(t),\ b(0)=1$

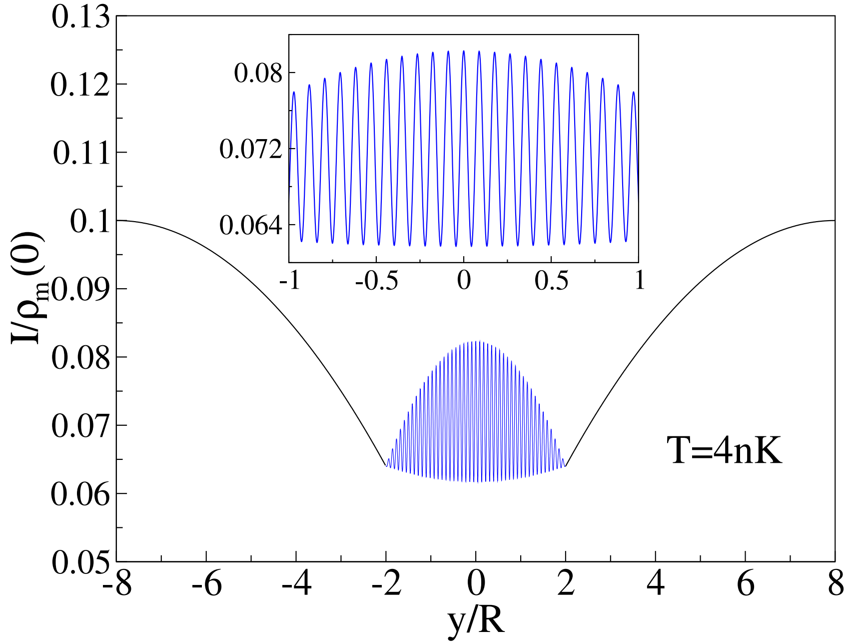
Öhberg & Santos, PRL 2002

Interference signal from density matrix

$$W(x,x',t)=\left\langle \Psi_B^*(x,t)\Psi_B(x',t)\right\rangle$$

TF limit: Interference signal





$\omega_x = 0.5kHz, \quad t = 16ms$

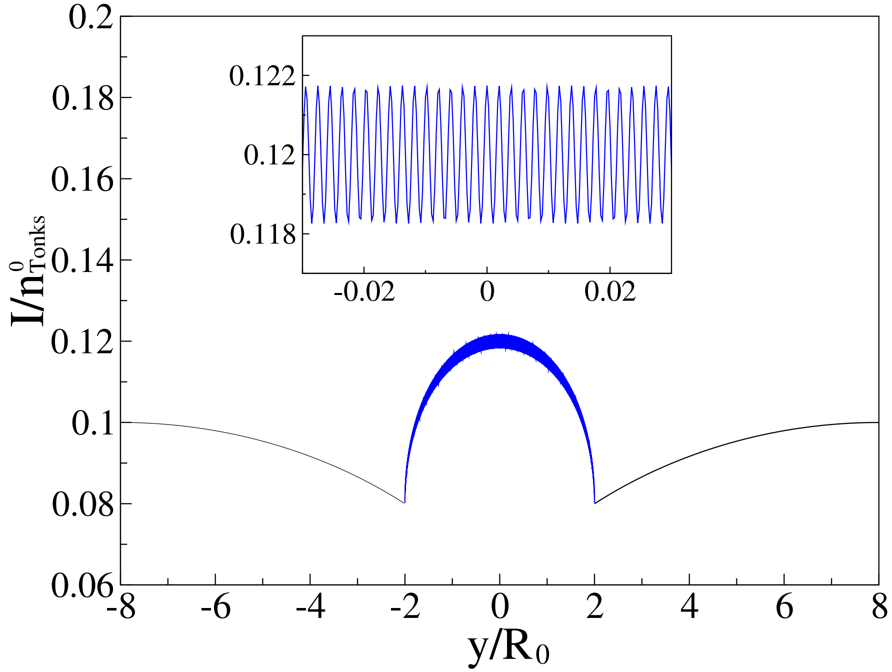
$\omega_x = 1kHz, \quad t = 8ms$

- Ring with $\omega_\perp = 50kHz$ and $1000\ {}^{23}Na$ atoms
- Circumference *16R* for TF radius *R*

Interference in Tonks-Girardeau limit

Chen & Egger, PRA 2003

- Interactions will decrease interference signal substantially compared to Thomas Fermi limit
- Big interaction effect
- Explicit confirmation from a fermionized picture possible *Das, Girardeau & Wright, PRL 2002*



$1000 \quad {}^{87}Rb$

$$\omega_\perp = 100kHz, \quad \omega_0 = 10Hz, \quad t = 1s$$

Disordered interacting bosons

- Field theory unstable for bosons
- So far only mean-field type approximate results, or numerical simulations
- Exact statements possible for 1D disordered bosons with strong repulsion:
  - Bose-Fermi mapping to free disordered fermions
  - Bose glass phase is mapped to Anderson localized fermionic phase

De Martino, Thorwart, Egger & Graham, cond-mat/0408xxx

Bose Hubbard model

- Bose Hubbard model in 1D

$$H = \sum_l \left(-J\left(b_{l+1}^* b_l + h.c.\right) + (h_l + bl^2) n_l + U n_l \left(n_l - 1\right)\right)$$

- Tunable on-site disorder $\langle h_l h_k \rangle_{dis} = \Delta_{dis} \delta_{lk}$
  - laser speckle pattern
  - incommensurate additional lattice
  - microchip-confined systems: Atom-surface interactions

Bose-Fermi Mapping

Consider hard-core bosons: $U \to \infty$

↷ only $n_l = 0,1$ possible!

Jordan-Wigner transformation to free fermions:

$$b_l = \exp\left(i\pi \sum_{j<l} c_j^* c_j \right) c_l$$

↷

$$H_F = \sum_l \left(-J\left(c_{l+1}^* c_l + h.c. \right) + (h_l + bl^2) c_l^* c_l \right)$$

Well known in clean case (Tonks-Girardeau), but also works with disorder!

Many-body boson wavefunction

N-boson wavefunction is Slater determinant of free fermion solutions $\psi_i(l)$ to single-particle energy ε_i

$$\Phi_\nu^B(l_1,\cdots,l_N) = \left(\prod_{i<j} \operatorname{sgn}(l_i - l_j)\right)\frac{1}{\sqrt{N!}}\det\left(\psi_i(l_j)\right)$$

$$E_\nu = \sum_{j=1}^{N} \varepsilon_i^{(j)}$$

Girardeau, J. Math. Phys. 1960

Physical observables

- All observables expressed by $\left|\Phi_\nu^B\right|^2$ are invariant under Bose-Fermi mapping, e.g. local density of states (LDoS)

$$\rho(\varepsilon, l) = \sum_{\nu} \sum_{l_2,\ldots,l_N} \delta(\varepsilon - E_\nu) \left|\Phi_\nu^B(l, l_2, \ldots, l_N)\right|^2$$

- Greatly simplified calculation for others, e.g. boson momentum distribution

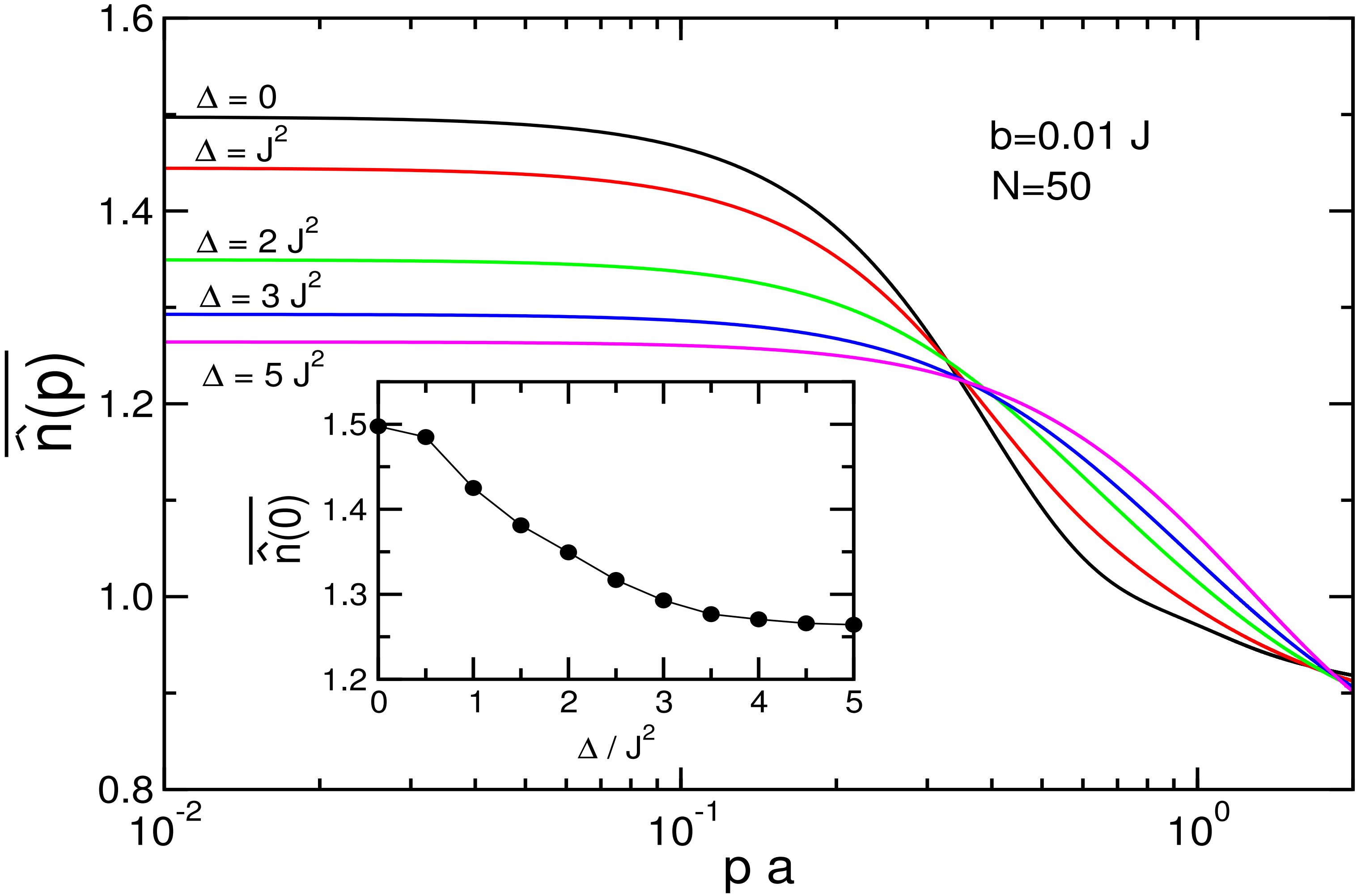
Boson momentum distribution

- Momentum distribution different for boson and fermion systems
- Bosonic one: $$\hat{n}(p) = \frac{1}{N} \sum_{ll'} e^{-ip(l-l')a} \left\langle b_l^* b_{l'} \right\rangle$$
- Jordan-Wigner transformation & Wick´s theorem give for fixed disorder:

$$\left\langle b_l^* b_{l'} \right\rangle = 2^{\,l-l'-1} \det G^{(l,l')}$$

$$G_{ij}^{(l,l')} = \left\langle c_{l'+i}^* c_{l'+j-1} \right\rangle - \frac{1}{2} \delta_{i,j-1}$$

Re



Numerically averaged over 300 disorder realizations, T=0

Continuum limit (homogeneous case)

- Low-energy expansion defines bispinor

$$c_l \approx \sqrt{a}\left[e^{ik_F x}\Psi_R(x) + e^{-ik_F x}\Psi_L(x)\right] \qquad x = la$$

- Free-fermion Hamiltonian $H = \int dx \Psi^* \hat{h} \Psi$

$$\hat{h} = -iv_F \sigma_z \partial_x + \mu(x) + \xi(x)\sigma_+ + \xi^*(x)\sigma_-$$

with $k_F = \pi N / L,\ v_F = k_F / m$

Disorder averages

- Disorder forward scattering can be eliminated by gauge transformation for incommensurate situation
- Backward scattering: $\left\langle \xi(x)\xi^*(x')\right\rangle_{dis} = \frac{v_F^2}{2\ell}\delta(x-x')$
- Consider weak disorder: $k_F \ell > 1$
- Standard free-fermion Hamiltonian for study of 1D Anderson localization, many results available (mainly via Berezinskii method)

LDoS distribution function

- Average DoS is simply $(\pi v_F)^{-1}$
- More interesting: Probability distribution of LDoS (normalized to average DoS)
- Closed sample: Regularization necessary, broadening η of sharp discrete energy levels
  - Inelastic processes, finite sample lifetime
- Result: Inverse Gaussian distribution

$$W(\rho) = \sqrt{\frac{4\eta\tau}{\pi\rho^3}} e^{-4\eta\tau(\rho-1)^2/\rho}$$

Al´tshuler & Prigodin, Zh.Eksp.Teor.Fis. 1989

Finite spatial resolution

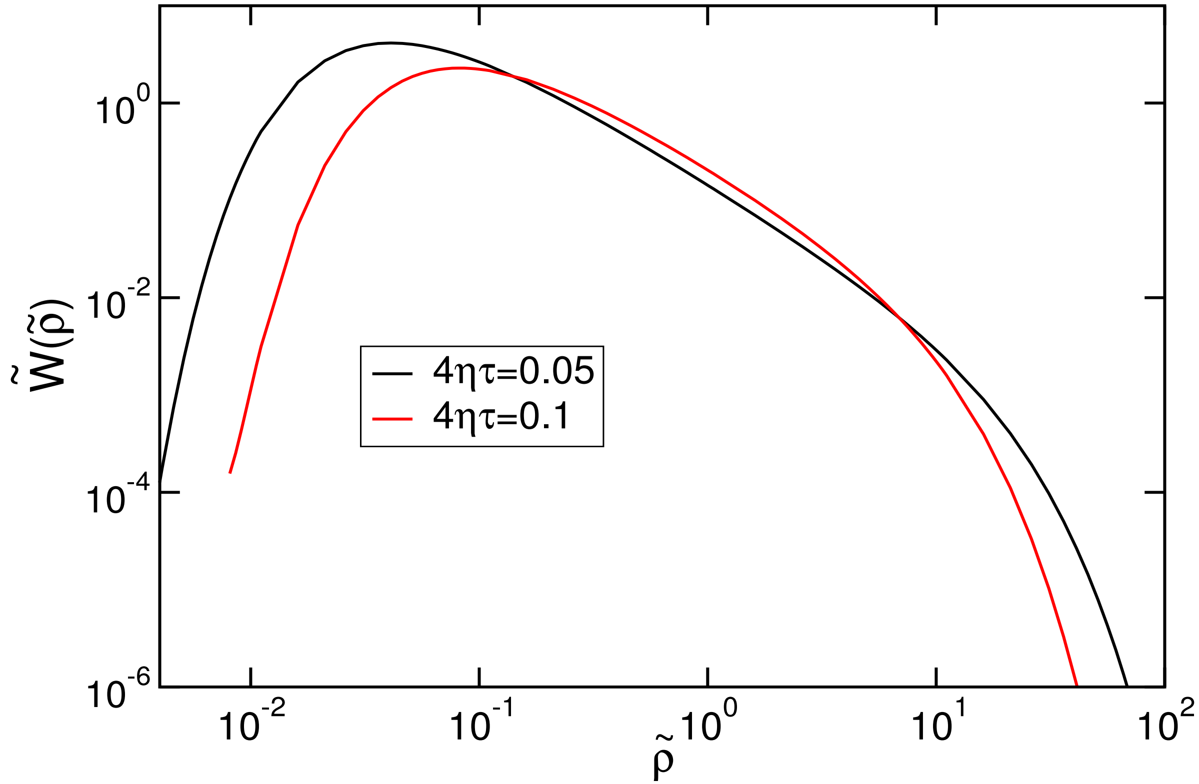
- LDoS can be measured using two-photon Bragg spectroscopy
- Finite spatial resolution (laser beam) in the range $k_F^{-1} < \delta < \ell$ defines smeared LDoS

$$\tilde{\rho}(\varepsilon, x) = \int_{-\delta/2}^{\delta/2} \frac{dy}{\delta} \rho(\varepsilon, x + y)$$

- Distribution function is then

$$\tilde{W}(\tilde{\rho}) = \frac{\eta\tau}{\pi} \int_4^{\infty} t dt \sin(\pi\eta\tau) \left(\frac{t+4}{t-4} \right)^{t\eta\tau} e^{-\eta\tau\tilde{\rho}\, t^2/2}$$

Implications



Anomalously small probability for small LDoS implies Poisson distribution of energetically close-by bosonic energy levels

No level repulsion as in Wigner-Dyson ensembles!

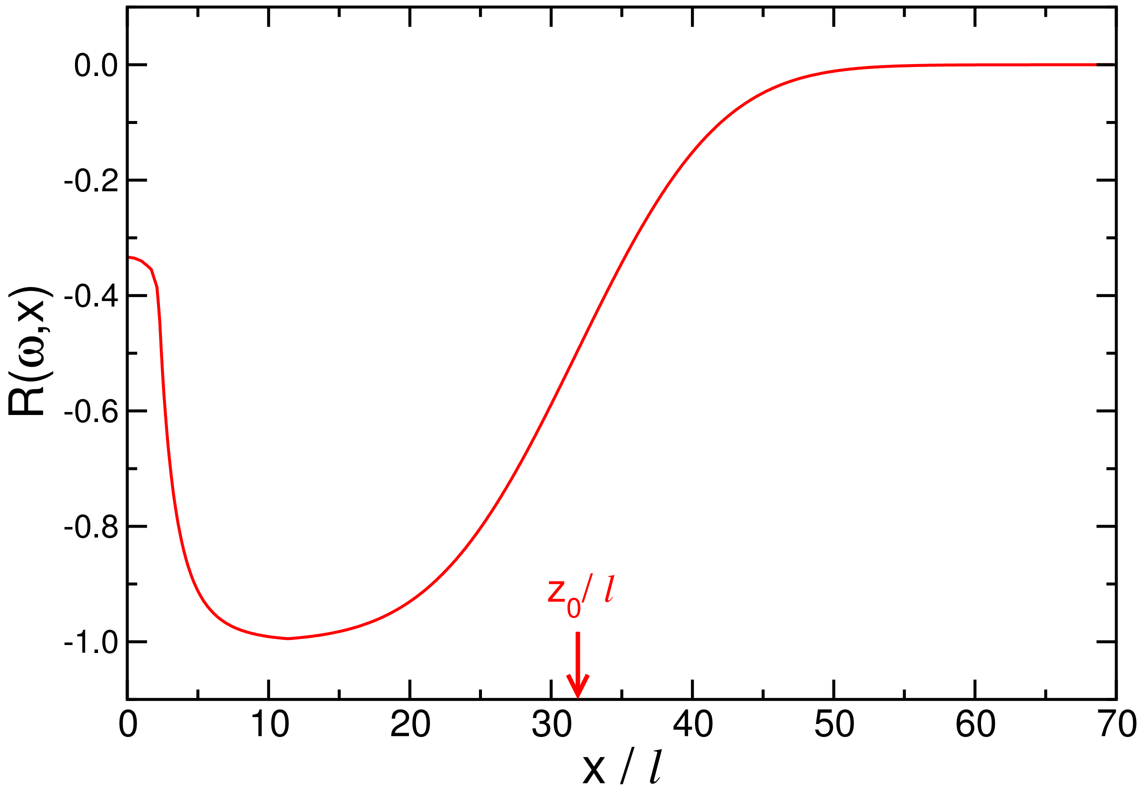
Spectral correlations

- LDoS correlations at different energies and locations

$$R(\Omega, x - x') = \langle \tilde{\rho}(\varepsilon, x)\tilde{\rho}(\varepsilon + \Omega, x') \rangle - 1$$

  - equals the fermionic correlator
  - consider low energies $\Omega\, \tau < 1$
- Limits: *Gorkov, Dorokhov & Prigara, Zh.Eksp.Teor.Fis.1983*
  - Large distances: uncorrelated value *R=0*
  - Short distance: *R* approaches *-1/3*
  - Deep dip at intermediate distances

Spectr



Deep dip for $\ell < |x - x| < z_0 = 2\ell \ln(/\Omega\tau)$

Then: $$R(\Omega, x) = \frac{1}{2}\left[erf\left(\frac{x - z_0}{2\sqrt{x z_0}} \right) - 1 \right]$$

Implications

- Energetically close-by states occupy
  - with high probability distant locations
  - but appreciable overlap at short distances
- Localized states are centered on many defects, complicated quantum interference phenomenon
- No Wigner-Dyson correlations, but Poisson statistics of uncorrelated energy levels

Other quantities

Mapping allows to extract many other experimentally relevant quantities:

- Compressibility, and hence sound velocity
- Density-density correlations, structure factor
- Time-dependent density profile (expansion)
  - crossover from short-time diffusion to long-time localization physics

Details and references:

De Martino, Thorwart, Egger & Graham, cond-mat/0408xxx

Replica field theory & Nonlinear σ model: Disordered interacting fermions

- Disorder average via replicas $\Psi_\alpha(\vec{r},\varepsilon),\ \alpha = 1,\ldots,n$

$$F = -k_B T \langle \ln Z \rangle$$

$$\ln Z = \lim_{n \to 0} \left(\frac{Z^n - 1}{n} \right)$$

- Disorder average ⟹ time-nonlocal four-fermion interactions, prefactor $\propto 1/\nu_0 \tau$

- Atom-atom or electron-electron interactions: Four-fermion interactions, e.g. pseudopotential, strength U_0

Towards the replica field theory

- Decouple disorder-induced four-fermion interactions via energy-bilocal field $Q_{\varepsilon\varepsilon'}^{\alpha\alpha'}(\vec{r})$
- Similar Hubbard-Stratonovich transformation to decouple interaction-induced four-fermion interactions via field $\Phi_{\alpha}(\vec{r},\omega)$
- Integrate out fermions
- Physics encoded in geometry of these fields, connection to theory of symmetric spaces
- Formally exact, includes nonperturbative effects

Saddle point structure

- Full action of replicated theory:

$$S[Q,\Phi]=\frac{\pi\nu_0}{4\tau}Tr\left(Q^2\right)+\frac{1}{2U_0}Tr\left(\Phi^2\right)-Tr\ln G^{-1}$$

$$G^{-1}=i\varepsilon-\varepsilon_p+i\Phi+\frac{i}{2\tau}Q$$

- Standard saddle point: vacuum of interacting disordered system („Fermi sea“)
  - Gauge transformation with linear functional *K*

$$Q = e^{iK[\Phi]}\Lambda e^{-iK[\Phi]}$$

$$\Lambda_{\varepsilon\varepsilon'}^{\alpha\alpha'}=\operatorname{sgn}(\varepsilon)\delta_{\alpha\alpha'}\delta_{\varepsilon\varepsilon'}$$

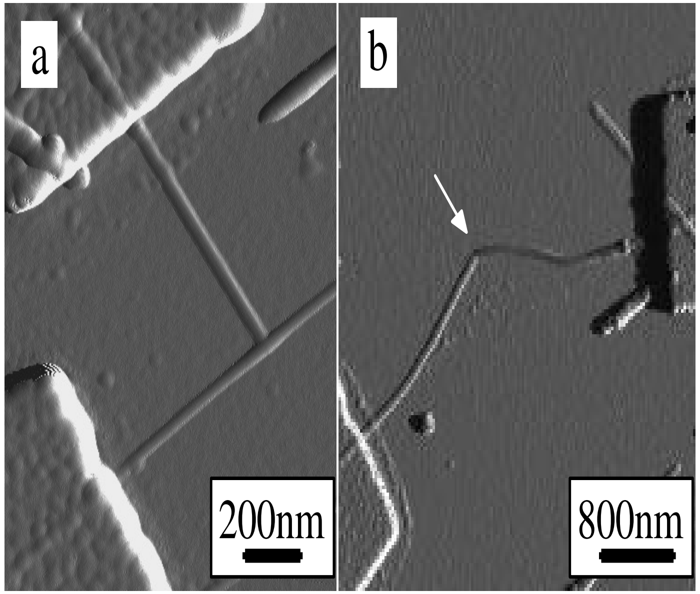
Nonlinear sigma model (NLσM)

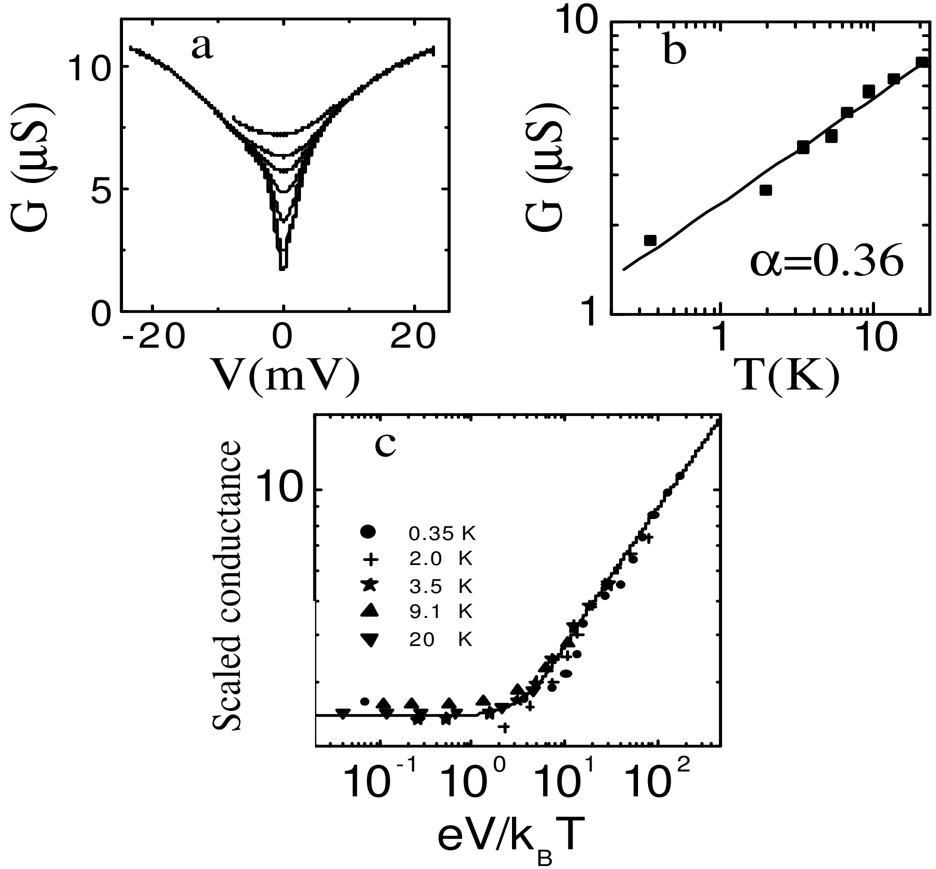
- Gradient expansion of logarithm for weak disorder and low energies gives interacting NLσM

  Finkel´stein, Zh. Eksp. Teor. Fiz. 1983

- Fluctuations around standard saddle give:
  - Diffuson (diffusively screened interaction)
  - Cooperon (weak localization)
  - Interaction corrections: Nonperturbative treatment of the zero-bias anomaly (ZBA)
- Caution: Large fluctuations (Q instantons) involving non-standard saddle points often important (e.g. for Wigner-Dyson spectral statistics)

Example: ZBA in multiwall nanotubes





Bachtold et al., PRL 2001

Pronounced non-Fermi liquid behavior

Diffusive interacting system: LDoS

Egger & Gogolin, PRL 2001, Chem. Phys. 2002

- Local (tunneling) density of states (LDoS)

$$\rho(x,\varepsilon) \propto \mathrm{Re} \int_0^\infty dt e^{i\varepsilon t/\hbar} \left\langle \Psi_F(x,t) \Psi_F^*(x,0) \right\rangle$$

  Microscopic nonperturbative theory:
  Interacting nonlinear σ model

- Nonperturbative result in interactions for LDoS, valid for diffusive multichannel wires

LDoS of interacting diffusive wire

LDoS ⟺ Debye-Waller factor *P(E)*:

$$\frac{\nu(E)}{\nu_0} = \int d\varepsilon \; P\left(E-\varepsilon\right)\frac{1+e^{-E/k_B T}}{1+e^{-\varepsilon/k_B T}}$$

$$P(E) = \mathrm{Re}\int_0^\infty \frac{dt}{\pi}\exp\left[iEt/\hbar + J\left(t\right)\right]$$

$$J(T=0,t) = \int_0^\infty \frac{d\omega}{\omega} I(\omega)\left(e^{-i\omega t}-1\right)$$

Connection to *P(E)* theory of Coulomb blockade

Spectral density: *P(E)* theory

- NLσM calculation gives for interaction U_0

$$I(\omega) = \frac{U_0}{2\pi(D^* - D)} \mathrm{Re} \sum_{n_\perp} \left(\left[-i\omega / D^* + n_\perp^2 / N^2 \right]^{-1/2} - \left(D^* \to D\right) \right)$$

$$D^* / D = 1 + \nu_0 U_0, \; D = u^2 \tau / 2$$

Field/particle diffusion constants

- For constant spectral density: Power law

$$\rho(\varepsilon) \propto \varepsilon^\alpha \quad \text{with} \quad \alpha = I(\omega \to 0)$$

Two-dimensional limit: Above the (transverse) Thouless energy

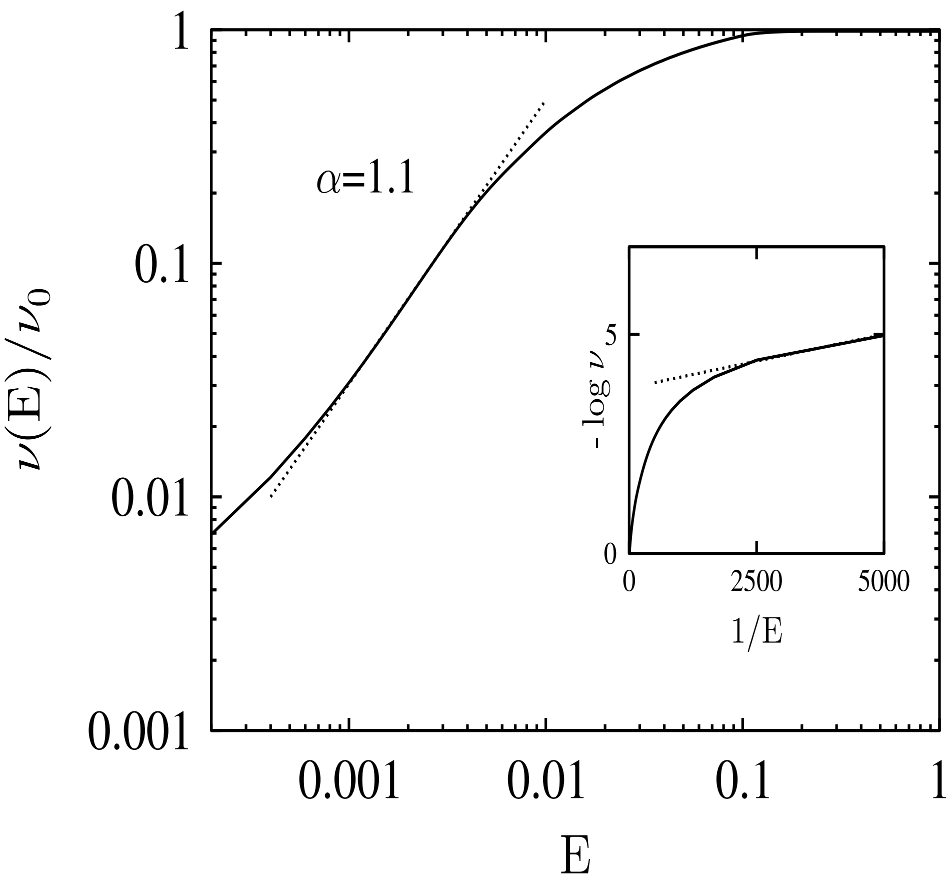
- For $E > E_c = \hbar D / L_\perp^2$, summation can be converted to integral, yields constant spectral density ➡ Power-law ZBA

$$\alpha = \frac{1}{2\pi \nu_0 D} \ln\left(D^* / D\right)$$

- Tunneling into interacting diffusive 2D system
- Logarithmic Altshuler-Aronov ZBA is exponentiated into power-law ZBA
- At low energies: Pseudogap behavior

Below the Thouless scale

- Apparent power law, like in experiment
- Smaller exponent for weaker interactions, only weak dependence on mean free path
- 1D pseudogap at very low energies
- Should also be observable for cold Fermionic atoms!



$$N = 10\,, U_0 / 2\pi u = 1$$

Conclusions

Concepts for noninteracting and interacting mesoscopic/atomic systems

- Luttinger liquid theory: Interference of 1D Bose matter waves
- Strongly interacting 1D bosons: Mapping to noninteracting fermions allows to apply solution of 1D Anderson localization
- Replica field theory, interacting nonlinear sigma model: TDoS of multichannel wires