Theoretical Quantum Optics and Quantum Information (SS 2021)

Exercise II (discussion: 05.05.2021)

1. *Coherent states*

   (a) EIGENSTATES OF $\hat{b}^\dagger$

   Show that an eigenstate of $\hat{b}^\dagger$ cannot exist.

   Hint: Expand a putative eigenstate of $\hat{b}^\dagger$ in the basis of Fock states $|n\rangle$

   $$|\beta\rangle = \sum_{n=0}^{\infty} |n\rangle \langle n|\beta\rangle \,.$$

   Then you will see that the creation operator produces a new superposition of occupations $|n\rangle$ with different range of occupancy than $|\beta\rangle$.

   (b) ACTION OF $\hat{b}^\dagger$

   Show that

   $$\hat{b}^\dagger |\alpha\rangle = \left(\frac{\partial}{\partial\alpha} + \frac{\alpha^*}{2}\right)|\alpha\rangle$$

   where $|\alpha\rangle$ is a coherent state, i.e. $\hat{b}\,|\alpha\rangle = \alpha\,|\alpha\rangle$. Use the Wirtinger derivative

   $$\frac{\partial}{\partial(x+iy)} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right).$$

   Note that the chain rule of differentiation does not hold in general for matrix functions!

   (c) SUPERPOSITION OF COHERENT STATES

   Calculate the probability distribution of the photon number and the mean photon number for the state

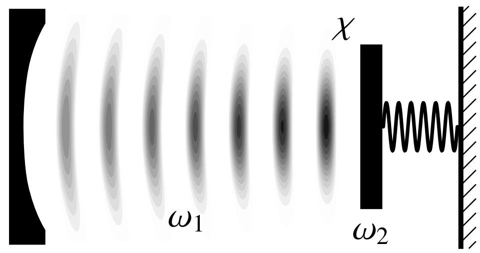
   $$|\psi_\pm\rangle \propto (|\alpha\rangle \pm |-\alpha\rangle), \tag{1}$$

   which is a superposition of two coherent states. Start by normalizing $|\psi_\pm\rangle$.

   Hint: The resulting prefactor can be simplified by the use of hyperbolic functions.

2. *Two coupled harmonic oscillators*

Consider a system that consists of two coupled harmonic oscillators with frequencies ω_1 and ω_2. The first oscillator corresponds to a mode of an electromagnetic field inside a cavity, which couples via radiation pressure to a mechanical oscillator, as shown in the figure. The parameter χ characterizes the coupling strength



between them. The dynamics of such a system is governed by the standard Hamiltonian in optomechanics

$$\hat{H} = \hbar \sum_{i=1}^{2} \omega_i \hat{b}_i^\dagger \hat{b}_i - \hbar\chi \hat{b}_1^\dagger \hat{b}_1 \left(\hat{b}_2^\dagger + \hat{b}_2\right) .$$

Here, $\hat{b}_i^\dagger$ and $\hat{b}_i$, $i \in \{1, 2\}$ denote the bosonic creation and annihilation operator of the corresponding harmonic oscillator.

(a) Show that the unitary displacement transformation

$$\hat{\mathcal{D}}^\dagger(\hat{\beta})\hat{H}\hat{\mathcal{D}}(\hat{\beta}) =: \hat{H}' \tag{2}$$

decouples the two harmonic oscillators for $\hat{\beta} = \frac{\chi}{\omega_2}\hat{b}_1^\dagger \hat{b}_1$. Here, the *generalized* displacement operator is defined as $\hat{\mathcal{D}}(\hat{\beta}) := \exp\left(\hat{\beta}\hat{b}_2^\dagger - \hat{\beta}^\dagger \hat{b}_2\right)$. Note that $\hat{\beta}$ is a hermitian operator, that fulfills $[\hat{\beta}, \hat{b}_1^\dagger \hat{b}_1] = 0$.

Hint: Verify first, that $\hat{\mathcal{D}}^\dagger(\hat{\beta})\hat{b}_2\hat{\mathcal{D}}(\hat{\beta}) = \hat{b}_2 + \hat{\beta}$.

(b) The displaced Hamiltonian $\hat{H}'$ obeys the following eigenequation

$$\hat{H}' \left|E_{n_1,n_2}\right\rangle = E_{n_1,n_2} \left|E_{n_1,n_2}\right\rangle .$$

Find the eigenenergies E_{n_1,n_2} and eigenstates $\hat{\mathcal{D}}\big(\hat{\beta}\big) \left|E_{n_1,n_2}\right\rangle$ of the non-displaced Hamiltonian $\hat{H}$.

(c) With the eigenstates

$$\hat{\mathcal{D}}(\hat{\beta})\,|E_{n_1,n_2}\rangle = \hat{\mathcal{D}}(\chi n_1/\omega_2)\,|n_1\rangle \otimes |n_2\rangle \equiv \hat{\mathcal{D}}(\chi n_1/\omega_2)\,|n_1, n_2\rangle \tag{3}$$

and the eigenenergies

$$E_{n_1,n_2} = \hbar\omega_1 n_1 + \hbar\omega_2 n_2 - \hbar\frac{\chi^2}{\omega_2} n_1^2 \tag{4}$$

one can study the dynamics of the system. Use the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\,|\psi(t)\rangle = \hat{H}\,|\psi(t)\rangle$$

to calculate the time evolution of the state $|\psi_0\rangle = |n_1\rangle \otimes |\alpha\rangle \equiv |n_1, \alpha\rangle$, where the first harmonic oscillator is initially prepared in a Fock state $|n_1\rangle$ and the second one in a coherent state $|\alpha\rangle$ by expanding $|\psi(t)\rangle$ in the eigenbasis of $\hat{H}$. Show that (up to a global phase)

$$|\psi(t)\rangle = |n_1\rangle \otimes \left|\eta(t) + \alpha e^{-i\omega_2 t}\right\rangle$$

with $\eta(t) := \frac{\chi n_1}{\omega_2}(1 - e^{-i\omega_2 t})$.

Hint: Use the expansion of a coherent state in the Fock basis $|\gamma\rangle = e^{-\frac{|\gamma|^2}{2}} \sum_{k=0}^{\infty} \frac{\gamma^k}{\sqrt{k!}}\,|k\rangle$ with $\gamma \in \mathbb{C}$. *Also use that* $\hat{\mathcal{D}}^\dagger(\mu)\,|\nu\rangle = \hat{\mathcal{D}}(-\mu)\,|\nu\rangle = |\nu - \mu\rangle$ with $\mu, \nu \in \mathbb{C}$, *up to a global phase.*

3. *Squeezed States*

(a) SQUEEZE OPERATOR

By using the Baker-Hausdorff formula,

$$e^{A}Be^{-A} = B + \frac{1}{1!}[A,B] + \frac{1}{2!}[A,[A,B]] + \frac{1}{3!}[A,[A,[A,B]]] + \cdots ,$$

show that

$$\hat{S}^\dagger(\varepsilon)\hat{b}\hat{S}(\varepsilon) = \hat{b}\cosh r - \hat{b}^\dagger e^{2i\phi}\sinh r$$
$$\hat{S}^\dagger(\varepsilon)\hat{b}^\dagger\hat{S}(\varepsilon) = \hat{b}^\dagger\cosh r - \hat{b}e^{-2i\phi}\sinh r$$

and

$$\hat{S}^\dagger(\varepsilon)f(\hat{b},\hat{b}^\dagger)\hat{S}(\varepsilon) = f(\hat{b}\cosh r - \hat{b}^\dagger e^{2i\phi}\sinh r\,,\, \hat{b}^\dagger\cosh r - \hat{b}e^{-2i\phi}\sinh r), \tag{5}$$

where

$$\hat{S}(\varepsilon) = \exp\left[\frac{1}{2}\left(\varepsilon^*\hat{b}^2 - \varepsilon(\hat{b}^\dagger)^2\right)\right], \quad \varepsilon = re^{2i\phi}$$

and $f(\hat{x},\hat{y})$ is a function which can be expanded into a power series.

(b) SQUEEZE AND DISPLACEMENT OPERATOR *(bonus)*

The order of the $\hat{D}$ and $\hat{S}$ operators in the definition of a squeezed state in Eq. (36) (see lecture) does matter, since

$$\hat{D}(\alpha)\hat{S}(\varepsilon) = \hat{S}(\varepsilon)\hat{D}(\gamma).$$

Express the parameter γ in terms of α and ε.

Hint: Use the transformation given in Eq. (5).

(c) PROPERTIES OF SQUEEZED STATES *(bonus)*

Calculate the mean photon number

$$\bar{n} = \langle\hat{n}\rangle = \langle\alpha,\varepsilon|\,\hat{n}\,|\alpha,\varepsilon\rangle = \langle\alpha,\varepsilon|\,\hat{b}^\dagger\hat{b}\,|\alpha,\varepsilon\rangle$$

and the variance of the photon probability distribution

$$(\Delta n)^2 = \langle\alpha,\varepsilon|\,\hat{n}^2\,|\alpha,\varepsilon\rangle - (\bar{n})^2$$

for the squeezed state $|\alpha,\varepsilon\rangle$.