

Theoretical Quantum Optics and Quantum Information (SS 2021)

Exercise I (discussion: 21.04.2021)

1. Density matrix

(a) VON NEUMANN EQUATION

Show that the density matrix

$$\rho(t) = |\Psi(t)\rangle \langle \Psi(t)|$$

obeys the equation

$$i\hbar \frac{d\rho(t)}{dt} = [\hat{H}, \rho(t)], \quad (1)$$

where \hat{H} is the Hamiltonian and $|\Psi(t)\rangle$ the state of the system.

(b) TRACE PRESERVATION

Use Eq. (1) to show that

$$\frac{d}{dt} \text{tr}(\rho(t)) = 0.$$

(c) HEISENBERG EQUATION

Derive the Heisenberg equation for an operator \hat{A} (with $\partial_t \hat{A} = 0$) in the Heisenberg picture, i.e.

$$i\hbar \frac{d\hat{A}(t)}{dt} = [\hat{A}(t), \hat{H}], \quad (2)$$

from Eq. (1) by starting with $\frac{d}{dt} \langle \hat{A} \rangle = \text{tr} \left(\frac{d\rho}{dt} \hat{A} \right)$ in the Schrödinger picture.

(d) SCHRÖDINGER VS. HEISENBERG

What is the difference between the Schrödinger picture (Eq. (1)) and the Heisenberg picture (Eq. (2))?

2. Hamiltonian of the electromagnetic field

(a) POLARISATION VECTORS

It is known from the lecture that the wave vector \vec{k} and the polarization vectors $\vec{\varepsilon}_{\vec{k}}^{(1)}$ and $\vec{\varepsilon}_{\vec{k}}^{(2)}$ form a right-handed system $(\vec{k}, \vec{\varepsilon}_{\vec{k}}^{(1)}, \vec{\varepsilon}_{\vec{k}}^{(2)})$ of orthogonal vectors. Show that for $-\vec{k}$ it is possible to choose

$$\begin{aligned} \vec{\varepsilon}_{-\vec{k}}^{(1)} &= \vec{\varepsilon}_{\vec{k}}^{(1)} \\ \text{and } \vec{\varepsilon}_{-\vec{k}}^{(2)} &= -\vec{\varepsilon}_{\vec{k}}^{(2)}. \end{aligned} \quad (3)$$

(b) CLASSICAL HAMILTONIAN

We consider the example of the electromagnetic field in a cube of size L (see lecture) with mode functions

$$\vec{u}_k(\vec{r}) = \frac{1}{L^{3/2}} \vec{\varepsilon}_{\vec{k}}^{(\lambda)} e^{i\vec{k} \cdot \vec{r}}. \quad (4)$$

Starting from the classical Hamilton function

$$H = \frac{1}{2} \int \left(\varepsilon_0 \vec{E}^2 + \mu_0 \vec{H}^2 \right) d^3\vec{r} \quad (5)$$

derive the equation

$$H = \sum_k \hbar \omega_k b_k^* b_k.$$

Remember that the vector potential can be expressed as

$$\vec{A}(\vec{r}, t) = \sum_k \left(\frac{\hbar}{2\omega_k \varepsilon_0} \right)^{\frac{1}{2}} (b_k(t) \vec{u}_k(\vec{r}) + b_k^*(t) \vec{u}_k^*(\vec{r})), \quad (6)$$

where $b_k(t) = b_k e^{-i\omega_k t}$ and $b_k^*(t) = b_k^* e^{i\omega_k t}$, that the electric field corresponds to $\vec{E}(\vec{r}, t) = -\frac{\partial \vec{A}(\vec{r}, t)}{\partial t}$ and the magnetic field is given by $\vec{B}(\vec{r}, t) = \vec{\nabla} \times \vec{A} = \mu_0 \vec{H}$. For the notation see the lecture.

Hint: Use the orthonormality of the mode functions $\int_V \vec{u}_k^*(\vec{r}) \vec{u}_{k'}(\vec{r}) d^3\vec{r} = \delta_{kk'}$. Use $\int_V \frac{1}{V} e^{\pm i(\vec{k} + \vec{k}') \cdot \vec{r}} d^3\vec{r} = \delta_{\vec{k}, -\vec{k}'}$ and the results from the previous question (Eq. (3)). For the electromagnetic wave it holds that $\int \varepsilon_0 \vec{E}^2 d^3\vec{r} = \int \mu_0 \vec{H}^2 d^3\vec{r}$.

(c) QUANTUM HAMILTONIAN

Now derive the Hamilton operator in the second quantization which is given by

$$\hat{H} = \frac{1}{2} \sum_k \hbar \omega_k \left(\hat{b}_k^\dagger \hat{b}_k + \hat{b}_k \hat{b}_k^\dagger \right). \quad (7)$$

For this purpose, replace in Eq. (6) b_k and b_k^* with the operators \hat{b}_k and \hat{b}_k^\dagger .

3. “Displacement” Operator

Show that

$$\hat{D}^\dagger(\alpha) \hat{b} \hat{D}(\alpha) = \hat{b} + \alpha,$$

$$\hat{D}^\dagger(\alpha) \hat{b}^\dagger \hat{D}(\alpha) = \hat{b}^\dagger + \alpha^*,$$

$$\hat{D}(\alpha + \beta) = \hat{D}(\alpha) \hat{D}(\beta) e^{-i \text{Im}(\alpha \beta^*)},$$

where

$$\hat{D}(\alpha) = \exp(\alpha \hat{b}^\dagger - \alpha^* \hat{b}).$$

Hint: Use the Baker-Campbell-Hausdorff identity, i.e. $e^{\hat{A} + \hat{B}} = e^{\hat{A}} e^{\hat{B}} e^{-\frac{1}{2} [\hat{A}, \hat{B}]}$ which holds if $[\hat{A}, [\hat{A}, \hat{B}]] = [\hat{B}, [\hat{A}, \hat{B}]] = 0$, and the Baker-Hausdorff formula

$$e^{\hat{A}} \hat{B} e^{-\hat{A}} = \hat{B} + \frac{1}{1!} [\hat{A}, \hat{B}] + \frac{1}{2!} [\hat{A}, [\hat{A}, \hat{B}]] + \frac{1}{3!} [\hat{A}, [\hat{A}, [\hat{A}, \hat{B}]]] + \dots$$

Requirements for the admission to the exam

- Exercise solutions do not have to be handed in. Before an exercise session, you send the number of the exercises that you have worked out and that you could present in the webex tutorial (without need for perfection!) via E-mail, to the following email address: **carrara@uni-duesseldorf.de**. In case that you do not manage to explain a marked exercise, no points will be given for the entire sheet! (These requirements may be adjusted...)
- For admission to the exam, you need:
 - 60% of the total exercises marked.
 - Present three (sub-) tasks in the webex tutorials.