1. **Von Neumann entropy**

Let $|\Psi\rangle$ be a two-qubit state given by

$$|\Psi\rangle = \frac{1}{\sqrt{2}} (|00\rangle + \cos \phi |10\rangle + \sin \phi |11\rangle).$$

Calculate the von Neumann entropy $S(\rho_a)$ of the reduced density matrix $\rho_a = \text{tr}_b (|\Psi\rangle\langle \Psi|)$ and graph the von Neumann entropy depending on $\cos \phi$.

2. **Measurements**

   (a) **Projectors**

   Let $|u\rangle$ and $|v\rangle$ be normalized vectors. Show that $|u\rangle \langle u|$ and $|v\rangle \langle v|$ are projectors. Moreover, show that $|u\rangle \langle u| + |v\rangle \langle v|$ is a projector iff $\langle u| v \rangle = 0$.

   Generalize this result to an arbitrary number of vectors.

   (b) **von Neumann Measurements**

   The measurement performed by a Stern-Gerlach apparatus oriented in z-direction is a set of orthogonal projectors, $P_1 = |0\rangle \langle 0|$ and $P_2 = |1\rangle \langle 1|$.

   (i) Show that this apparatus cannot distinguish between $|\psi_1\rangle := \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle)$ and $\rho_2 := \frac{1}{2} |0\rangle \langle 0| + \frac{1}{2} |1\rangle \langle 1|.$

   (ii) Devise a measurement that distinguishes between $|\psi_1\rangle$ and $\rho_2$. 
3. **Minimum error discrimination**

Suppose that a source creates one of the two non-orthogonal states

\[ |u\rangle = \cos \alpha |0\rangle + \sin \alpha |1\rangle, \]
\[ |v\rangle = \sin \alpha |0\rangle + \cos \alpha |1\rangle \]

with corresponding probabilities \( p_u \) and \( p_v \) such that \( p_u + p_v = 1 \). The aim of the *minimum error discrimination* is to distinguish the states \( |u\rangle \) and \( |v\rangle \) with a two-element POVM \( \{F_u, F_v\} \), where the measurement outcomes \( u \) and \( v \) are associated with the states \( |u\rangle \) and \( |v\rangle \), respectively. The POVM elements \( F_u \) and \( F_v \) are chosen such that the probability of making an error is minimal.

(a) Using the completeness condition \( F_u + F_v = 1 \), show that the probability of making an error is given by

\[
P_{\text{err}} = p_u \langle u | F_v | u \rangle + p_v \langle v | F_u | v \rangle
\]
\[= p_u - \text{Tr} \left[ AF_u \right],\]

where the operator \( A \) is given by

\[ A = p_u |u\rangle \langle u| - p_v |v\rangle \langle v|.\]

(b) It can be shown that \( P_{\text{err}} \) is minimal, when \( F_u \) is chosen as \( F_u = |\psi\rangle \langle \psi| \), where \( |\psi\rangle \) is the eigenvector of the operator \( A \) to its maximal eigenvalue. In this case, \( \text{Tr}[AF_u] = \lambda_{\text{max}} \), where \( \lambda_{\text{max}} \) is the maximal eigenvalue of \( A \). Compute the minimal \( P_{\text{err}} \) via calculating the eigenvalues of \( A \) as a function of the product \( |\langle u | v \rangle|^2 \).
4. **Unambiguous state discrimination**

Let

\[ |u\rangle = \cos \alpha |0\rangle + \sin \alpha |1\rangle \]
\[ |v\rangle = \sin \alpha |0\rangle + \cos \alpha |1\rangle \]

be two non-orthogonal states and consider the POVM

\[ F_1 = \frac{1 - |v\rangle\langle v|}{1 + \langle u | v \rangle}, \quad F_2 = \frac{1 - |u\rangle\langle u|}{1 + \langle v | u \rangle}, \quad F_3 = 1 - F_1 - F_2. \]

(a) Describe the following properties of the POVM:

(i) Are the POVM elements orthogonal?

(ii) Are they projectors?

(iii) Which rank do they have?

(b) Show that this POVM is suitable to do so-called **unambiguous state discrimination** between \(|u\rangle\) and \(|v\rangle\). This means, that

\[ F_1 \] detects the state \(|u\rangle\) and \(\text{tr} (F_1 |v\rangle\langle v|) = 0\),

\[ F_2 \] detects the state \(|v\rangle\) and \(\text{tr} (F_2 |u\rangle\langle u|) = 0\) and

\[ F_3 \] gives an inconclusive answer.

5. **Entanglement swapping** (bonus)

In the lecture the entanglement swapping protocol was presented. Show that a Bell measurement on particles Z and A in the state

\[ |\psi^{\text{total}}\rangle = |\psi\rangle_{YZ} \otimes |\psi^{-}\rangle_{AB} \]
\[ = [\alpha |01\rangle + \beta |10\rangle]_{YZ} \otimes \frac{1}{\sqrt{2}} [|01\rangle - |10\rangle]_{AB}, \]

followed by an appropriate rotation of the particles B, results in the desired state

\[ |\psi\rangle_{YB} = [\alpha |01\rangle + \beta |10\rangle]_{YB}, \]

up to a global phase. Which rotation of B is needed for which outcome of the Bell measurement?