1. **Shannon entropy:**

   (a) **Relative entropy**

   Use the definition of the relative entropy,
   \[
   H(p(x)||q(x)) := \sum_x p(x)(\log_2 p(x) - \log_2 q(x)),
   \]

   to show that this quantity is always non-negative,
   \[
   H(p(x)||q(x)) \geq 0,
   \]

   where equality holds iff \( p(x) = q(x) \) \( \forall x \).

   *Hint:* Use \( \ln x \leq x - 1 \) \( \forall x \in \mathbb{R}^+ \). Alternatively, use Jensen’s inequality \( f(\langle X \rangle) \leq \langle f(X) \rangle \) for a convex function \( f \) and random variable \( X \).

   (b) **Sub-additivity of the entropy**

   Show that the Shannon entropy is sub-additive

   \[
   H(X,Y) \leq H(X) + H(Y).
   \]

   *Hint:* Use the positivity of the relative entropy for \( H(p(x,y)||p(x)p(y)) \).

   (c) **Mutual information**

   Recall the definition of the mutual information,

   \[
   I(X : Y) := H(X) - H(X|Y).
   \]

   It measures the amount of information that one learns about \( X \), when one gets to know \( Y \). Show that this quantity is non-negative and that it vanishes iff \( X \) and \( Y \) are independent.
2. **Single qubit systems**

(a) **Bloch vector of pure states**
Show that all pure states of the single qubit density operator \( \hat{\rho}_{\text{pure}} \) are represented by a Bloch vector \( \vec{s} \) of length one.

*Hint:* Use \( \hat{\sigma}_i \hat{\sigma}_j = \delta_{ij} \mathbb{1} + i\epsilon_{ijk} \hat{\sigma}_k \).

(b) **Bloch vectors of orthogonal states**
Which relation do the Bloch vectors \( \vec{s}_1 \) and \( \vec{s}_2 \) of two orthogonal single qubit density operators (i.e. \( \text{tr}(\hat{\rho}_1 \hat{\rho}_2) = 0 \)) have? Can orthogonal qubit density operators be mixed?

(c) **Mixed states**
Prove that the decomposition of a mixed state density matrix into a weighted sum of projectors is not unique. (Give an example of two different weighted sums of projectors onto pure states, leading to the same mixed density operator.)
3. Entanglement

Consider the composite Hilbert space $\mathcal{H}_{ab} = \mathcal{H}_a \otimes \mathcal{H}_b$, where $\mathcal{H}_a$ and $\mathcal{H}_b$ denote Hilbert spaces associated to a single qubit.

(a) Schmidt decomposition

Find the Schmidt ranks of the two states

$$ |\Psi_1\rangle = \frac{1}{2} \left( |00\rangle - i |01\rangle - i |10\rangle - |11\rangle \right), $$

$$ |\Psi_2\rangle = \frac{1}{2} \left( |00\rangle + i |01\rangle + i |10\rangle + |11\rangle \right), $$

with $|\Psi_1\rangle, |\Psi_2\rangle \in \mathcal{H}_{ab}$. Are these states separable or entangled?

(b) Local unitary transformation

Show that a local unitary transformation, i.e. $U_{ab} = U_a \otimes U_b$ applied to a product state leads to a product state.

(c) Partial trace and entanglement

Apply the unitary transformation

$$ U_{cNOT} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix} $$

to the product state

$$ \hat{\rho}_{ab} = |\Psi\rangle \langle \Psi| = \frac{1}{2} \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1
\end{pmatrix}, $$

with $|\Psi\rangle = \frac{1}{\sqrt{2}} \left( |0\rangle + |1\rangle \right) \otimes |1\rangle$, to get $\hat{\rho}_{ab}' = U_{cNOT} \hat{\rho}_{ab} U_{cNOT}^\dagger$. Calculate $\hat{\rho}_a' = \text{tr}_b(\hat{\rho}_{ab}')$ and describe the properties of that state. Is $\hat{\rho}_{ab}'$ separable or entangled? Comment on the result in consideration of 3b.