1. **K-partite entanglement of pure states**

   (a) Given a pure state $|\psi_{ABC}\rangle \in \mathcal{H}^A \otimes \mathcal{H}^B \otimes \mathcal{H}^C$, with $\dim(\mathcal{H}^A) = \dim(\mathcal{H}^B) = \dim(\mathcal{H}^C) = 2$. Assume that $|\psi_{ABC}\rangle$ is $A - BC$ and $AB - C$ separable. Show that $|\psi_{ABC}\rangle$ is $A - B - C$ separable.

   (b) Show that if a state $|\psi\rangle$ is $k$-separable for all possible $k$-partite splits, $|\psi\rangle$ is $(k + 1)$-separable for all $(k + 1)$-partite splits. If a state $|\psi\rangle$ is biseparable for all possible bipartitions, is it also fully separable? (Hints: use the result of 1a)

2. **Partial transpose criterion**

   (a) Calculate the eigenvalues of the partial transpose of the quantum state

   $$\rho_p = (1 - p)|\phi^\perp\rangle\langle\phi^\perp| + \frac{p}{4} \mathbb{1} \quad \text{with} \quad |\phi^\perp\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) .$$

   For which range $p$ with $0 \leq p \leq 1$ is this state entangled?

   (b) Consider $\rho_{AB}, \sigma_{AB} \in B(\mathcal{H}_A \otimes \mathcal{H}_B)$ and verify the following properties:

   i. $\rho^\Gamma_{AB} = (\rho^\Gamma_A)^\Gamma_B$,

   ii. $\text{tr}(\rho^\Gamma_{AB}) = \text{tr}(\rho^\Gamma_{AB})$ and

   iii. $\text{tr}(\rho^\Gamma_{AB} \sigma_{AB}) = \text{tr}(\rho_{AB} \sigma^\Gamma_{AB})$.

   (c) (**BONUS**) Show that

   i. $\rho^\Gamma_{AB} \geq 0 \Leftrightarrow \rho^\Gamma_{AB} \geq 0,$

   ii. $\left[(\mathbb{1}_A \otimes B) \rho_{AB} (\mathbb{1}_A \otimes B^\dagger) \right]^\Gamma_A = (\mathbb{1}_A \otimes B) \rho_{AB}^\Gamma (\mathbb{1}_A \otimes B^\dagger)$ and

   iii. $\rho^\Gamma_{AB} \geq 0 \Rightarrow \left[(A \otimes B) \rho_{AB} (A^\dagger \otimes B^\dagger) \right]^\Gamma_A \geq 0.$
3. **ENTANGLEMENT**

Consider the density operator

$$
\rho_{ABC} = \frac{1}{N} \left[ 2|GHZ\rangle\langle GHZ| + \sum_{x=1}^{3} a_x |k_x\rangle\langle k_x| + \frac{1}{a_x} |\bar{k}_x\rangle\langle \bar{k}_x| \right]
$$

with \( k \in \{k_1 = 001, k_2 = 010, k_3 = 100\} \), \( \bar{k} \) being the binary representation which is given by inverting all digits of \( k \), \( |GHZ\rangle = \frac{1}{\sqrt{2}} (|000\rangle + |111\rangle) \) and \( \prod_{x=1}^{3} a_x \neq 1 \)

(a) Calculate the normalization \( N \) and write \( \rho_{ABC} \) as a matrix in the computational basis. What are the restrictions on \( a_x \) such that \( \rho_{ABC} \) is positive semidefinite?

(b) Show that the state has a positive partial transpose with respect to every bipartite split.

(Hint: Make use of the block structure of \( \rho_{ABC}^{\Gamma_I} \), where \( \Gamma_I \) denotes the partial transposition with respect to the subsystems \( I \in \{A, B, C\} \)).

(c) (*BONUS*) Show that the state is separable with respect to every bipartite split.

(Hint: Again use the block structure and that in 2 \times 2\)-dimensions the PPT criterion is necessary and sufficient for separability).