1. *P* and Wigner functions

(a) **An useful Integral**

Show that for \( \text{Re}(\lambda) > 0 \) and arbitrary \( \mu, \nu, \eta \in \mathbb{C} \)

\[
\frac{1}{\pi} \int d^2\eta \ e^{-\lambda|\eta|^2+\mu\eta+\nu\eta^*} = \frac{1}{\lambda} \ e^{\frac{\mu\nu}{\lambda}}
\]

holds. **Hint:** \( \int_{-\infty}^{\infty} e^{-x^2} = \sqrt{\pi} \).

(b) **Gaussian convolution**

Show that the Wigner function is a Gaussian convolution of the \( P \) function, i.e.

\[
W (\alpha) = \frac{2}{\pi} \int P (\beta) \exp \left[ -2 |\beta - \alpha|^2 \right] d^2\beta.
\]

(c) **Characteristic function of a Fock state**

Calculate the characteristic function of a Fock state \( |n\rangle \), i.e.

\[
\text{tr} \left[ |n\rangle \langle n| \exp \left( \eta \hat{b}^\dagger - \eta^* \hat{b} \right) \right].
\]

**Hint:** \( L_n(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{k!} x^k \) is a Laguerre polynomial.

(d) **Wigner function of a Fock state**

Evaluate the Wigner function for the Fock state \( |n=1\rangle \) and show that it can be negative.

(e) **Representation of the Wigner function (bonus)**

Show that the Wigner function in suitable variables

\[
\alpha = \sqrt{\frac{\omega}{2\hbar}} x + \frac{i}{\sqrt{2\hbar\omega}} p
\]

can be represented as

\[
W (\alpha) = \frac{1}{\pi} \int \langle x - \frac{y}{2} | \hat{\rho} | x + \frac{y}{2} \rangle \exp \left( \frac{iyp}{\hbar} \right) dy.
\]

**Hint:** Express \( \hat{b} \) and \( \hat{b}^\dagger \) in terms of position and momentum operators. Show that \( \exp \left[ -\frac{i\hbar a}{\hbar} \right] |x\rangle = |x+a\rangle \) and use that \( \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp (iax) \ dx = \delta (a) \) and that \( \int_{-\infty}^{\infty} f(x) \delta (x-a) \ dx = f (a) \).
2. Angular momentum and the Jaynes-Cummings Model

(a) **Angular Momentum Operators**

Show that

\[
\hat{J}_x = \frac{\hat{b}^\dagger \hat{\sigma}_- + \hat{b} \hat{\sigma}_+}{2\sqrt{\hat{N}}}, \quad \hat{J}_y = \frac{i(\hat{b}^\dagger \hat{\sigma}_- - \hat{b} \hat{\sigma}_+)}{2\sqrt{\hat{N}}} \quad \text{and} \quad \hat{J}_z = \hat{\sigma}_z \text{ with}
\]

\[
\hat{\sigma}_z = \frac{\hat{\sigma}_+ \hat{\sigma}_- - \hat{\sigma}_- \hat{\sigma}_+}{2}, \quad \hat{N} = \hat{b}^\dagger \hat{b} + \hat{\sigma}_+ \hat{\sigma}_- - \hat{\sigma}_- \hat{\sigma}_+ + \frac{1}{2} \mathbb{1},
\]

obey the algebra of the angular momentum operators, i.e. show that

i. \([\hat{J}_x, \hat{J}_y] = i\hat{J}_z,\)

ii. \([\hat{J}_y, \hat{J}_z] = i\hat{J}_x \text{ and}\)

iii. \([\hat{J}_z, \hat{J}_x] = i\hat{J}_y.\)

(*Hint:* Note that \([\hat{N}, \hat{J}_i] = 0, \) remember the relations \([b, b^\dagger] = \mathbb{1}\) and \(\{\sigma_+, \sigma_-\} = \mathbb{1}.\)

(b) **Conserved Quantity \(\hat{N}\) (bonus)**

Show that \(\hat{N}\) is a conserved quantity, i.e. that \([\hat{N}, \hat{H}] = 0,\) where \(\hat{H}\) is given by

\[
\hat{H} = \hbar \omega_0 \hat{\sigma}_z + \hbar \omega \hat{b}^\dagger \hat{b} + \hbar g(\hat{b}^\dagger \hat{\sigma}_- + \hat{b} \hat{\sigma}_+).
\]

(c) **Spin in an Effective Magnetic Field**

Show that the Jaynes-Cummings model Hamiltonian (Eq. (1)) describes a spin in an effective magnetic field, i.e. find a \(\mathbf{B}\) such that

\[
\hat{H} = \hat{\mathbf{J}} \cdot \mathbf{B} + c,
\]

where the vector \(\hat{\mathbf{J}}\) contains the angular momentum operators and \(c\) is an arbitrary energy shift.

(d) **Spectrum of the Hamiltonian**

Find the eigenvalues of the Hamiltonian \(\hat{H}\) using Eq. (1).

Note, consistent with the lecture we use: \(\hat{\sigma}_z = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.\)