1. **Shannon entropy:**

   (a) **Relative entropy**

   Use the definition of the relative entropy,

   \[ H(p(x)||q(x)) := \sum_x p(x)(\log_2 p(x) - \log_2 q(x)), \]

   to show that this quantity is always non-negative,

   \[ H(p(x)||q(x)) \geq 0, \]

   where equality holds iff \( p(x) = q(x) \ \forall x. \)

   **Hint:** Use that \( \ln x < x - 1 \) for \( x > 0, x \neq 1 \) and \( \ln x = x - 1 \) for \( x = 1. \) Alternatively, use Jensen’s inequality ( \( f(\langle X \rangle) \leq \langle f(X) \rangle \) for a convex function \( f \) and random variable \( X \).)

   (b) **Sub-additivity of the entropy**

   Show that the Shannon entropy is sub-additive

   \[ H(X,Y) \leq H(X) + H(Y). \]

   **Hint:** Use the positivity of the relative entropy for \( H(p(x,y)||p(x)p(y)). \)

   (c) **Mutual information**

   Recall the definition of the mutual information,

   \[ I(X : Y) := H(X) - H(X|Y). \]

   It measures the amount of information that is learnt about \( X \) when one gets to know \( Y. \) Show that this quantity is non-negative and that it vanishes iff \( X \) and \( Y \) are independent.
2. Single qubit systems

(a) **Bloch vector of pure states**

Show that all pure states of the single qubit density operator \( \rho_{\text{pure}} \) are represented by a Bloch vector \( \vec{s} \) of length one.

(b) **Bloch vectors of orthogonal states**

Which relation do the Bloch vectors \( \vec{s}_1 \) and \( \vec{s}_2 \) of two orthogonal single qubit density operators (i.e. \( \text{tr}(\rho_1 \rho_2) = 0 \)) have? Can orthogonal qubit density operators be mixed?

*Hint:* Use \( \sigma_i \sigma_j = \delta_{ij} \mathbb{1} + i \epsilon_{ijk} \sigma_k \) and \( \text{tr}(\sigma_i) = 0 \).

(c) **Mixed states**

Prove that the decomposition of a mixed state density matrix into a weighted sum of projectors is not unique. (Give an example of two different weighted sums of projectors onto pure states, leading to the same mixed density operator.)

3. Entanglement

(a) **Schmidt decomposition**

Find the Schmidt ranks of the two states

\[
|\Psi_1\rangle = \frac{1}{2} \left( |00\rangle - i |01\rangle - i |10\rangle - |11\rangle \right),
\]

\[
|\Psi_2\rangle = \frac{1}{2} \left( |00\rangle + i |01\rangle + i |10\rangle + |11\rangle \right).
\]

Are the states separable or entangled?

(b) **Local unitary transformation**

Show that a local unitary transformation, i.e. \( U_{ab} = U_a \otimes U_b \) applied to a product state leads to a product state.

(c) **Partial trace and entanglement**

Apply the unitary transformation

\[
U_{\text{cNOT}} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
\end{pmatrix}
\]
to the product state

\[
\rho_{ab} = |\Psi\rangle\langle \Psi| = \frac{1}{2} \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1
\end{pmatrix},
\]

with \(|\Psi\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle)|1\rangle\), to get

\[
\rho'_{ab} = U_{c\text{NOT}} \rho_{ab} U_{c\text{NOT}}^\dagger.
\]

Calculate \(\rho'_a = \text{tr}_b(\rho'_{ab})\) and describe the properties of that state. Is \(\rho'_{ab}\) separable or entangled? Comment on the result in consideration of 3b.