1. **P and Wigner functions**

(a) **AN USEFUL INTEGRAL**

Show that for \(\text{Re}(\lambda) > 0\) and arbitrary \(\mu, \nu, \eta \in \mathbb{C}\)

\[
\frac{1}{\pi} \int d^2\eta \ e^{-\lambda|\eta|^2 + \mu\eta + \nu\eta^*} = \frac{1}{\lambda} e^{\frac{\mu\nu}{\lambda}}
\]

holds. **Hint:** \(\int_{-\infty}^{\infty} e^{-x^2} = \sqrt{\pi}\).

(b) **GAUSSIAN CONVOLUTION**

Show that the Wigner function is a Gaussian convolution of the \(P\) function, i.e.

\[
W(\alpha) = \frac{2}{\pi} \int P(\beta) \exp\left[-2|\beta - \alpha|^2\right] d^2\beta.
\]

(c) **CHARACTERISTIC FUNCTION OF A FOCK STATE**

Calculate the characteristic function of a Fock state \(|n\rangle\), i.e.

\[
\text{tr}\left[|n\rangle \langle n| \exp\left(\eta \hat{b}^\dagger - \eta^* \hat{b}\right)\right].
\]

**Hint:** \(L_n(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{k!} x^k\) is a Laguerre polynomial.

(d) **WIGNER FUNCTION OF A FOCK STATE**

Evaluate the Wigner function for the Fock state \(|n = 1\rangle\) and show that it can be negative.

(e) **REPRESENTATION OF THE WIGNER FUNCTION (bonus)**

Show that the Wigner function in suitable variables

\[
\alpha = \sqrt{\frac{\omega}{2\hbar}} x + \frac{i}{\sqrt{2\hbar\omega}} p
\]

can be represented as

\[
W(\alpha) = \frac{1}{\pi} \int \langle x - \frac{y}{2} | \hat{\rho} | x + \frac{y}{2} \rangle \exp\left(\frac{ipy}{\hbar}\right) dy.
\]

**Hint:** Express \(\hat{b}\) and \(\hat{b}^\dagger\) in terms of position and momentum operators. Show that \(\exp\left[-\frac{i\alpha}{\hbar}\right] |x\rangle = |x + a\rangle\) and use that \(\frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(iax) \ dx = \delta(a)\) and that \(\int_{-\infty}^{\infty} f(x) \delta(x - a) \ dx = f(a)\).
2. Angular momentum and the Jaynes-Cummings Model

(a) Angular momentum operators

Show that
\[ \hat{J}_x = \frac{(\hat{b}^\dagger \hat{\sigma}_+ + \hat{b} \hat{\sigma}_-)}{2\sqrt{\hat{N}}} \], \[ \hat{J}_y = i\frac{\hat{b}^\dagger \hat{\sigma}_- - \hat{b} \hat{\sigma}_+}{2\sqrt{\hat{N}}} \] and \( \hat{J}_z = \hat{\sigma}_z \) with \( \hat{\sigma}_z = \hat{\sigma}_+ + \hat{\sigma}_- \), \( \hat{N} = \hat{b}^\dagger \hat{b} + \hat{\sigma}_+ \hat{\sigma}_- = \hat{b}^\dagger \hat{b} + \hat{\sigma}_z + \frac{1}{2} \mathbb{1} \),

obey the algebra of the angular momentum operators, i.e. show that

i. \([\hat{J}_x, \hat{J}_y]\) = \(i\hat{J}_z\),

ii. \([\hat{J}_y, \hat{J}_z]\) = \(i\hat{J}_x\) and

iii. \([\hat{J}_z, \hat{J}_x]\) = \(i\hat{J}_y\).

(Hint: Note that \([\hat{N}, \hat{J}_i]\) = 0, remember the relations \([b, b^\dagger] = \mathbb{1}\) and \(\{\sigma_+, \sigma_-\} = \mathbb{1}\).)

(b) Conserved quantity \(\hat{N}\) (bonus)

Show that \(\hat{N}\) is a conserved quantity, i.e. that \([\hat{N}, \hat{H}] = 0\), where \(\hat{H}\) is given by
\[ \hat{H} = \hbar \omega_0 \hat{\sigma}_z + \hbar \omega \hat{b}^\dagger \hat{b} + \hbar g (\hat{b}^\dagger \hat{\sigma}_- + \hat{b} \hat{\sigma}_+) \].

(1)

(c) Spin in an effective magnetic field

Show that the Jaynes-Cummings model Hamiltonian (Eq. (1)) describes a spin in an effective magnetic field, i.e. find a \(\mathbf{B}\) such that
\[ \hat{H} = \hat{\mathbf{J}} \cdot \mathbf{B} + c, \]

where the vector \(\hat{\mathbf{J}}\) contains the angular momentum operators and \(c\) is an arbitrary energy shift.

(d) Spectrum of the Hamiltonian

Find the eigenvalues of the Hamiltonian \(\hat{H}\) using Eq. (1).

Note, consistent with the lecture we use: \[ \hat{\sigma}_z = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \].