1. Parameter estimation and Standard State Tomography

Bob received 120 copies of a single qubit quantum state $\rho$ from Alice. Now he intends to reconstruct the state using the following POVM

$$E_1 = \frac{1}{3} |0\rangle\langle 0| = \frac{1}{3} + \sigma_z,$$

$$E_2 = \frac{1}{3} |+\rangle\langle +| = \frac{1}{3} + \sigma_x,$$

$$E_3 = \frac{1}{3} |i\rangle\langle i| = \frac{1}{3} + \sigma_y,$$

$$E_4 = 1 - E_1 - E_2 - E_3,$$

where $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$, $|i\rangle = (|0\rangle + i|1\rangle)/\sqrt{2}$.

(a) [Parameter estimation]
Please calculate the 'radius' $\zeta$ of the set $\Gamma_\zeta = \{ \rho' : ||P_n - P||_1 \leq \zeta \}$, if Bob expects an estimation failure probability $\varepsilon = 0.01$. How can Bob decrease the 'radius' of $\Gamma_\zeta$ to 0.1 without increasing the failure probability $\varepsilon$?
(HINT: For $n \gg 1$, $\ln(n+1) \approx \ln n$. The solution for $e^{ax+b} = \ln x$ and $x > 1$ is $x = -\frac{1}{a} W_1 (-\frac{a}{e^b})$, where $W_1(z)$ is the lower branch of Lambert’s function.)

(b) [Standard state tomography]
After measuring the 120 copies with the POVM, Bob gets the following table

<table>
<thead>
<tr>
<th>Number of $E_n$ outcome</th>
<th>$E_1$</th>
<th>$E_2$</th>
<th>$E_3$</th>
<th>$E_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>8</td>
<td>36</td>
<td>20</td>
<td>56</td>
</tr>
</tbody>
</table>

Reconstruct the state $\rho$ using standard state tomography.

(c) [Statistical error]
Let $\vec{R} = (\rho_{00}, \text{Re} \rho_{01}, \text{Im} \rho_{01}, \rho_{11})^T$ and Bob expects only $\varepsilon = 0.01$ failure probability in his state tomography. Estimate the distance from the reconstructed vector $\vec{R}^{(n=120)}$ to the real one $\vec{R}$, i.e. upper bound the 1-norm $||\vec{R}^{(n)} - \vec{R}||_1$.

Is the number of copies $n = 120$ sufficient for a standard state tomography with $\varepsilon = 0.01$?
(HINT: for vector $\vec{v}$ and matrix $M$ it holds $||M : \vec{v}||_1 \leq ||M||_1 ||\vec{v}||_1$, where $|| \cdot ||_1$ is 1-norm with definition $||M||_1 = \max_j \sum_i |m_{ij}|$, $||\vec{v}||_1 = \sum_i |v_i|$.)