1. *K*-partite entanglement of pure states

(a) Given a pure state $|\psi_{ABC}\rangle \in \mathcal{H}^A \otimes \mathcal{H}^B \otimes \mathcal{H}^C$, with $\dim(\mathcal{H}^A) = \dim(\mathcal{H}^B) = \dim(\mathcal{H}^C) = 2$. Assume that $|\psi_{ABC}\rangle$ is $A-BC$ and $AB-C$ separable. Show that $|\psi_{ABC}\rangle$ is $A-B-C$ separable.

(b) Show that if a state $|\psi\rangle$ is $k$-separable for all possible $k$-partite splits, $|\psi\rangle$ is $(k+1)$-separable for all $(k+1)$-partite splits. If a state $|\psi\rangle$ is biseparable for all possible bipartitions, is it also fully separable? (Hints: use the result of 1a)

2. Partial transpose criterion

(a) Calculate the eigenvalues of the partial transpose of the quantum state

$$\rho_p = (1 - p)|\phi^+\rangle\langle\phi^+| + \frac{p}{4} \mathbb{1} \text{ with } |\phi^+\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle).$$

For which range $p$ with $0 \leq p \leq 1$ is this state entangled?

(b) Consider $\rho_{AB}, \sigma_{AB} \in B(\mathcal{H}_A \otimes \mathcal{H}_B)$ and verify the following properties:

i. $\rho_{AB}^T = (\rho_{AB}^A)^\Gamma_B$

ii. $\text{tr} (\rho_{AB}^A) = \text{tr} (\rho_{AB}^B)$

(Hint: first show that $\text{tr}_A(\rho_{AB}^A) = \text{tr}_A(\rho_{AB})$)

iii. $\text{tr} (\rho_{AB}^A \sigma_{AB}) = \text{tr} (\rho_{AB} \sigma_{AB})$

(c) (*BONUS*) Show that

i. $\rho_{AB}^A \geq 0 \iff \rho_{AB}^B \geq 0,$

ii. $[(\mathbb{1}_A \otimes B) \rho_{AB} (\mathbb{1}_A \otimes B^\dagger)]^A = (\mathbb{1}_A \otimes B) \rho_{AB}^A (\mathbb{1}_A \otimes B^\dagger)$ and

iii. $\rho_{AB}^A \geq 0 \Rightarrow [(A \otimes B) \rho_{AB} (A^\dagger \otimes B^\dagger)]^A \geq 0.$
3. Entanglement

Consider the density operator

$$\rho_{ABC} = \frac{1}{N} \left[ 2|GHZ\rangle\langle GHZ| + \sum_{x=1}^{3} a_x |k_x\rangle\langle k_x| + \frac{1}{a_x} |\bar{k}_x\rangle\langle \bar{k}_x| \right]$$

with $k \in \{k_1 = 001, k_2 = 010, k_3 = 100\}$, $\bar{k}$ being the binary representation which is given by inverting all digits of $k$, $|GHZ\rangle = \frac{1}{\sqrt{2}} (|000\rangle + |111\rangle)$ and $\prod_{x=1}^{3} a_x \neq 1$

(a) Calculate the normalization $N$ and write $\rho_{ABC}$ as a matrix in the computational basis. What are the restrictions on $a_x$ such that $\rho_{ABC}$ is positive?

(b) Show that the states have a positive partial transpose with respect to every bipartite split.

(Hint: Make use of the block structure of $\rho_{ABC}^{\Gamma_I}$, where $\Gamma_I$ denotes the partial transposition with respect to the subsystems $I \in \{A, B, C\}$).

(c) (*BONUS*) Show that the states are separable with respect to every bipartite split.

(Hint: Again use the block structure and that in 2 x 2-dimensions the PPT criterion is necessary and sufficient for separability).