1. K-partite entanglement of pure states

(a) Given a pure state $|\psi_{ABC}\rangle \in \mathcal{H}^A \otimes \mathcal{H}^B \otimes \mathcal{H}^C$, with $\dim(\mathcal{H}^A) = \dim(\mathcal{H}^B) = \dim(\mathcal{H}^C) = 2$. Assume that $|\psi_{ABC}\rangle$ is $A-BC$ and $AB-C$ separable. Show that $|\psi_{ABC}\rangle$ is $A-B-C$ separable.

(b) If a pure state $|\psi\rangle$ is $k$-separable for all possible $k$-partite splits, show that $|\psi\rangle$ is $(k+1)$-separable for all $(k+1)$-partite splits. If a pure state $|\psi\rangle$ is biseparable for all possible bipartitions, is it also fully separable? (Hint: use the result of 1a)

2. Partial transpose criterion

(a) Calculate the eigenvalues of the partially transposed quantum state $\rho_p = (1 - p)|\phi^+\rangle\langle\phi^+| + \frac{p}{4} \mathbb{1}$ with $|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.

For which range $p$ with $0 \leq p \leq 1$ is this state entangled?

(b) Consider $\rho_{AB}, \sigma_{AB} \in B(\mathcal{H}_A \otimes \mathcal{H}_B)$ and verify the following properties:

i. $\rho_{AB}^T = (\rho_{AB}^\Gamma)^\tau_B$,

ii. $\text{tr}(\rho_{AB}^\Gamma) = \text{tr}(\rho_{AB}^\Gamma)$

(Hint: first show that $\text{tr}_A(\rho_{AB}^\Gamma) = \text{tr}(\rho_{AB})$)

iii. $\text{tr}(\rho_{AB}^\Gamma \sigma_{AB}) = \text{tr}(\rho_{AB} \sigma_{AB}^\Gamma)$

(c) (*BONUS*) Show that

i. $\rho_{AB}^\Gamma \geq 0 \iff \rho_{AB}^\Gamma \geq 0$,

ii. $[(1_A \otimes B) \rho_{AB} (1_A \otimes B^\dagger)]^\Gamma_A = (1_A \otimes B) \rho_{AB}^\Gamma (1_A \otimes B^\dagger)$ and

iii. $\rho_{AB}^\Gamma \geq 0 \Rightarrow [(A \otimes B) \rho_{AB} (A^\dagger \otimes B^\dagger)]^\Gamma_A \geq 0$. 

3. **Entanglement**

Consider the density operator

\[
\rho_{ABC} = \frac{1}{N} \left[ 2|GHZ\rangle\langle GHZ| + \sum_{x=1}^{3} a_x |k_x\rangle\langle k_x| + \frac{1}{a_x} |\bar{k}_x\rangle\langle \bar{k}_x| \right]
\]

with \( k \in \{k_1 = 001, k_2 = 010, k_3 = 100\}, \bar{k} \) being the binary representation which is given by inverting all digits of \( k \), \(|GHZ\rangle = \frac{1}{\sqrt{2}} (|000\rangle + |111\rangle)\) and \( \prod_{x=1}^{3} a_x \neq 1 \)

(a) Calculate the normalization \( N \) and write \( \rho_{ABC} \) as a matrix in the computational basis. What are the restrictions on \( a_x \) such that \( \rho_{ABC} \) is positive?

(b) Show that the states have a positive partial transpose with respect to every bipartite split.

(Hint: Make use of the block structure of \( \rho_{ABC}^{\Gamma_I} \), where \( \Gamma_I \) denotes the partial transposition with respect to the subsystems \( I \in \{A, B, C\} \)).

(c) (*BONUS*) Show that the states are separable with respect to every bipartite split.

(Hint: Again use the block structure and that in 2 \times 2\)-dimensions the PPT criterion is necessary and sufficient for separability).