1. \textit{P and Wigner functions}

(a) \textbf{Integral}

Remember the rules for integration in the complex plane by showing that for \( \text{Re}(\lambda) > 0 \) and arbitrary \( \mu, \nu, \eta \in \mathbb{C} \)
\[
\frac{1}{\pi} \int d^2 \eta \, e^{-\lambda|\eta|^2 + \mu \eta + \nu \eta^*} = \frac{1}{\lambda} \, e^{\frac{\mu \nu}{\lambda}}
\]
holds. (This identity is useful in the following questions.)

(b) \textbf{Gaussian convolution}

Show that the Wigner function is a Gaussian convolution of the \( P \) function, i.e.
\[
W(\alpha) = \frac{2}{\pi} \int P(\beta) \exp \left[ -2 |\beta - \alpha|^2 \right] d^2 \beta.
\]

(c) \textbf{Characteristic function of a Fock state}

Calculate the characteristic function of a Fock state \( |n\rangle \), i.e.
\[
\text{tr} \left[ |n\rangle \langle n| \exp \left( \eta \hat{b}^\dagger - \eta^* \hat{b} \right) \right].
\]

(d) \textbf{Wigner function of a Fock state}

Evaluate the Wigner function for the Fock state \( |n = 1\rangle \) and show that it can be negative.

(e) \textbf{Representation of the Wigner function (bonus)}

Show that the Wigner function in suitable variables
\[
\alpha = \sqrt{\frac{\omega}{2\hbar}} x + \frac{i}{\sqrt{2\hbar \omega}} p
\]
can be represented as
\[
W(\alpha) = \frac{1}{\pi} \int \langle x - \frac{y}{2}\mid \hat{p}\mid x + \frac{y}{2}\rangle \exp \left( \frac{i py}{\hbar} \right) dy.
\]

\textit{Hint:} Express \( \hat{b} \) and \( \hat{b}^\dagger \) in terms of position and momentum operators. Show that \( \exp \left[ -\frac{ix}{2} \right] |x\rangle = |x + a\rangle \) and use that \( \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp (iax) \, dx = \delta(a) \) and that \( \int_{-\infty}^{\infty} f(x) \delta(x-a) \, dx = f(a) \).
2. Angular momentum and the Jaynes-Cummings Model

(a) Angular momentum operators

Show that
\[
\hat{J}_x = \frac{(\hat{b}^\dagger \hat{\sigma}_- + \hat{b} \hat{\sigma}_+)}{2\sqrt{\hat{N}}}, \quad \hat{J}_y = \frac{i(\hat{b}^\dagger \hat{\sigma}_- - \hat{b} \hat{\sigma}_+)}{2\sqrt{\hat{N}}}, \quad \text{and} \quad \hat{J}_z = \hat{\sigma}_z
\]
with
\[
\hat{\sigma}_z = \frac{\hat{\sigma}_+ \hat{\sigma}_- - \hat{\sigma}_- \hat{\sigma}_+}{2}, \quad \hat{N} = \hat{b}^\dagger \hat{b} + \hat{\sigma}_+ \hat{\sigma}_- = \hat{b}^\dagger \hat{b} + \hat{\sigma}_z + \frac{1}{2},
\]

obey the algebra of the angular momentum operators, i.e. show that

i. \([\hat{J}_x, \hat{J}_y]\) = \(i\hat{J}_z\),

ii. \([\hat{J}_y, \hat{J}_z]\) = \(i\hat{J}_x\) and

iii. \([\hat{J}_z, \hat{J}_x]\) = \(i\hat{J}_y\).

(Hint: Note that \([\hat{N}, \hat{J}_i]\) = 0, remember the relations \([\hat{b}, \hat{b}^\dagger]\) = 1 and \{\(\hat{\sigma}_+, \hat{\sigma}_-\} = 1\).

(b) Conserved quantity \(\hat{N}\) (bonus)

Show that \(\hat{N}\) is a conserved quantity, i.e. that \([\hat{N}, \hat{H}] = 0\), where \(\hat{H}\) is given by
\[
\hat{H} = \hbar \omega_0 \hat{\sigma}_z + \hbar \omega \hat{b}^\dagger \hat{b} + \hbar g (\hat{b}^\dagger \hat{\sigma}_- + \hat{b} \hat{\sigma}_+).
\]

(1)

(c) Spin in an effective magnetic field

Show that the Jaynes-Cummings model Hamiltonian (Eq. (2)) describes a
spin in an effective magnetic field, i.e. find \(\mathbf{B}\) such that
\[
\hat{H} = \hat{\mathbf{J}} \cdot \mathbf{B} + c,
\]

(2)

where the components of the vector \(\hat{\mathbf{J}}\) are given by equation (1) and \(c\) is an
arbitrary energy shift.

(d) Spectrum of the Hamiltonian

Find the eigenvalues of the Hamiltonian \(\hat{H}\) using Eq. (2).

Note, consistent with the lecture we use:
\[
\hat{\sigma}_z = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]