1. **Von Neumann entropy**

Let $|\Psi\rangle$ be a two-qubit state given by

$$|\Psi\rangle = \frac{1}{\sqrt{2}} \left( |00\rangle + \cos \phi |10\rangle + \sin \phi |11\rangle \right).$$

Calculate the von Neumann entropy of the reduced density matrix $S(\rho_a)$, where $\rho_a = \text{tr}_b (|\Psi\rangle \langle \Psi|)$ and graph the von Neumann entropy depending on $\cos \phi$.

2. **Measurements**

(a) **Projectors**

Let $|u\rangle$ and $|v\rangle$ be normalized vectors. Show that $|u\rangle \langle u|$ and $|v\rangle \langle v|$ are projectors. Moreover, show that $|u\rangle \langle u| + |v\rangle \langle v|$ is a projector iff $\langle u | v \rangle = 0$.

Generalize this result to an arbitrary number of vectors.

(b) **von Neumann measurements**

The measurement performed by a Stern-Gerlach apparatus oriented in z-direction is a set of orthogonal projectors, $P_1 = |0\rangle \langle 0|$ and $P_2 = |1\rangle \langle 1|$.

(i) Show that this apparatus cannot distinguish between $|\psi_1\rangle := \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle)$ and $\rho_2 := \frac{1}{2} |0\rangle \langle 0| + \frac{1}{2} |1\rangle \langle 1|$.

(ii) Devise a measurement that distinguishes between $|\psi_1\rangle$ and $\rho_2$. 
3. Unambiguous state discrimination

Let

\[ |u\rangle = \cos \alpha |0\rangle + \sin \alpha |1\rangle \]

\[ |v\rangle = \sin \alpha |0\rangle + \cos \alpha |1\rangle \]

be two non-orthogonal states and let

\[ F_1 = \frac{1 - |v\rangle\langle v|}{1 + |u\rangle\langle u|}, \]

\[ F_2 = \frac{1 - |u\rangle\langle u|}{1 + |v\rangle\langle v|}, \]

\[ F_3 = 1 - F_1 - F_2 \]

be a POVM.

(a) Describe the following properties of the POVM:

(i) Are the POVM elements orthogonal?

(ii) Are they projectors?

(iii) Which rank do they have?

(b) Show that this POVM is suitable to do so-called unambiguous state discrimination between \(|u\rangle\) and \(|v\rangle\). This means, that

\[ F_1 \] detects the state \(|u\rangle\) and \(\text{tr}(F_1 |v\rangle\langle v|) = 0\),

\[ F_2 \] detects the state \(|v\rangle\) and \(\text{tr}(F_2 |u\rangle\langle u|) = 0\) and

\[ F_3 \] gives an inconclusive answer.

4. Entanglement swapping

In the lecture the entanglement swapping protocol was presented. Show that a Bell measurement on particles Z and A in the state

\[ |\psi^{\text{total}}\rangle = |\psi\rangle_{YZ} \otimes |\psi^-\rangle_{AB} \]

\[ = [\alpha |01\rangle + \beta |10\rangle]_{YZ} \otimes \frac{1}{\sqrt{2}} [||01\rangle - |10\rangle]_{AB}, \]

followed by an appropriate rotation of the particles B, results in the desired state \(|\psi\rangle_{BY}\), up to a global phase. Which rotation of B is needed for which outcome of the Bell measurement?