1. Shannon’s noiseless coding theorem.

Consider an alphabet composed of five letters \{e, t, f, o, i\} and two languages where the frequency of the letters are given by:

<table>
<thead>
<tr>
<th>Letter</th>
<th>Language 1</th>
<th>Language 2</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Prob. distribution $p_1$</td>
<td>Prob. distribution $p_2$</td>
</tr>
<tr>
<td>a</td>
<td>0.3</td>
<td>0.07</td>
</tr>
<tr>
<td>t</td>
<td>0.21</td>
<td>0.01</td>
</tr>
<tr>
<td>f</td>
<td>0.2</td>
<td>0.65</td>
</tr>
<tr>
<td>o</td>
<td>0.18</td>
<td>0.15</td>
</tr>
<tr>
<td>i</td>
<td>0.11</td>
<td>0.12</td>
</tr>
</tbody>
</table>

(a) Calculate the average minimum number of bits asymptotically needed to encode a letter in the two languages of the table above.

(b) Consider the following encoding:

\[
a \rightarrow 0001 \quad t \rightarrow 001 \quad f \rightarrow 00001 \quad o \rightarrow 00000 \quad i \rightarrow 1.
\]

What is the average number of bits used for encoding a letter in the two languages?

(c) Create for each language a new encoding which uses on average less bits than the one proposed above and calculate the average number of bits required by your encoding.

(d) Consider the following encoding:

\[
a \rightarrow 0 \quad t \rightarrow 1 \quad f \rightarrow 00 \quad o \rightarrow 11 \quad i \rightarrow 10.
\]

Calculate the average number of bits needed for encoding a letter in language 1. This encoding would need less bits than an optimal encoding (see [1]). What is the problem with this encoding?
2. Single qubit systems

(a) Bloch vector of pure states
Show that all pure states of the single qubit density operator \( \rho_{\text{pure}} \) are represented by a Bloch vector \( \vec{s} \) with length one.

(b) Bloch vectors of orthogonal states
Which relation do the Bloch vectors \( \vec{s}_1 \) and \( \vec{s}_2 \) of two orthogonal single qubit density operators (i.e. \( \text{tr}(\rho_1 \rho_2) = 0 \)) have? Can orthogonal qubit density operators be mixed?
(Hint: use \( \sigma_i \sigma_j = \delta_{ij} \mathbb{1} + i \epsilon_{ijk} \sigma_k \) and \( \text{tr}(\sigma_i) = 0 \))

(c) Mixed states
Prove that the decomposition of a mixed state density matrix into a weighted sum of projectors is not unique. (Give an example of two different weighted sums of projectors onto pure states, leading to the same mixed density operator.)
3. Entanglement

(a) Schmidt decomposition

Find the Schmidt ranks of the two states

\[ |\Psi_1\rangle = \frac{1}{\sqrt{2}} (|00\rangle - i |01\rangle - i |10\rangle - |11\rangle), \]

\[ |\Psi_2\rangle = \frac{1}{\sqrt{2}} (|00\rangle + i |01\rangle + i |10\rangle + |11\rangle). \]

Are the states separable or entangled?

(b) Local unitary transformation

Show that a local unitary transformation, i.e. \( U_{ab} = U_a \otimes U_b \) applied to a product state leads to a product state.

(c) Partial trace and entanglement

Apply the unitary transformation

\[ U_{\text{cNOT}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \]

to the product state

\[ \rho_{ab} = |\Psi\rangle\langle\Psi| = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \]

with \( |\Psi\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) |1\rangle \), to get \( \rho'_{ab} = U_{\text{cNOT}} \rho_{ab} U_{\text{cNOT}}^\dagger \). Calculate \( \rho'_a = \text{tr}_b(\rho'_{ab}) \) and describe the properties of that state. Is \( \rho'_{ab} \) separable or entangled? Comment on the result in consideration of \( 3b \).