1. Coherent states

(a) Eigenstates of $\hat{b}^\dagger$

Show that an eigenstate of $\hat{b}^\dagger$ cannot exist.

*Hint:* Expand an eigenstate of $\hat{b}^\dagger$ in the basis of Fock states $|n\rangle$

$$|\beta\rangle = \sum_{n=0}^{\infty} |n\rangle \langle n| \beta\rangle.$$  

Then you will see that the creation operator produces a new superposition of occupations $|n\rangle$ with different range of occupancy than $|\beta\rangle$.

(b) Action of $\hat{b}^\dagger$

Show that

$$\hat{b}^\dagger |\alpha\rangle = \left(\frac{\partial}{\partial \alpha} + \frac{\alpha^*}{2}\right) |\alpha\rangle$$

where $|\alpha\rangle$ is a coherent state, i.e. $\hat{b} |\alpha\rangle = \alpha |\alpha\rangle$.

*Hint:* For $z, z^* \in \mathbb{C}$, the partial derivative $\frac{\partial}{\partial z}$ of a complex function $f(z, z^*)$ treats the variable $z^*$ as a constant.

(c) Evolution of the coherent state

Solve the Schrödinger equation to obtain evolution of the coherent state $|\alpha\rangle$.

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle.$$  

*Hint:* Solve the Schrödinger equation for Fock states

$$i\hbar \frac{\partial}{\partial t} |n\rangle = \hbar \omega \left(\hat{b}^\dagger \hat{b} + \frac{1}{2}\right) |n\rangle.$$
2. Coherent phase states

The state of the classical oscillator can be described either in terms of its quadrature components $x$ and $p$, or in terms of the amplitude and phase, so that $x + ip = A \exp(i\phi)$. In classical mechanics one can introduce the action and angle variables, which have the same Poisson brackets as the coordinate and momentum. However, in the quantum case we meet serious mathematical difficulties trying to define the phase operator in such a way that the commutation relation $[\hat{n}, \hat{\phi}] = i$ would be fulfilled, where $\hat{n} = \hat{b}^\dagger \hat{b}$. One can introduce the exponential phase operators

$$e^{i\hat{\phi}} = \hat{E}_- = (\hat{b}^\dagger \hat{b} + 1)^{-1/2} \hat{b} \quad \text{and} \quad e^{-i\hat{\phi}} = \hat{E}_+ = \hat{b}^\dagger (\hat{b}^\dagger \hat{b} + 1)^{-1/2}.$$

(a) Eigenstates of $\hat{E}_-$

Show that the normalizable state

$$|\varepsilon\rangle = \sqrt{1 - |\varepsilon|^2} \sum_{n=0}^{\infty} \varepsilon^n |n\rangle, \quad |\varepsilon| < 1$$

(1)

is an eigenstate of the operator $\hat{E}_-$.  

(b) Probability of the $n$th Fock state

Show that the pure quantum state (eq. 1) has the same probability distribution $|\langle n | \varepsilon \rangle|^2$ as the mixed thermal state described by the density operator

$$\hat{\rho} = \frac{1}{1 + \bar{n}} \sum_{n=0}^{\infty} \left( \frac{\bar{n}}{1 + \bar{n}} \right)^n |n\rangle \langle n|$$

if one identifies the mean photon number $\bar{n} \equiv \langle n \rangle$ with $\frac{|\varepsilon|^2}{1 - |\varepsilon|^2}$.

(c) Classical states

Show that for a coherent state $|\alpha\rangle$, where $\alpha = |\alpha| \exp(i\theta)$, the approximations

$$\langle \alpha | \cos \hat{\phi} | \alpha \rangle \approx \cos \theta \quad \text{and} \quad \langle \alpha | \sin \hat{\phi} | \alpha \rangle \approx \sin \theta$$

hold for large $|\alpha| \gg 1$. Thus the coherent state $|\alpha\rangle$ behaves like a classical wave with phase $\theta$. Hint:

$$\langle \alpha | \cos \hat{\phi} | \alpha \rangle = \langle 0 | \hat{D}^\dagger(\alpha) \left( \frac{(\hat{b}^\dagger \hat{b} + 1)^{-1/2} + \hat{b}^\dagger (\hat{b}^\dagger \hat{b} + 1)^{-1/2}}{2} \right) \hat{D}(\alpha) |0\rangle$$

$$= \langle 0 | \left( (\hat{b}^\dagger + \alpha^*)(\hat{b} + \alpha) + 1 \right)^{-1/2} (\hat{b} + \alpha) + (\hat{b}^\dagger + \alpha^*)(\hat{b} + \alpha) + 1 \right)^{-1/2} |0\rangle$$

$$\approx \langle 0 | \frac{(\hat{b} + \alpha) + (\hat{b}^\dagger + \alpha^*)}{2 |\alpha|} |0\rangle \quad \text{for large} \quad |\alpha| \gg 1.$$
3. Squeezed States

(a) Squeeze Operator

By using the Baker-Hausdorff formula,

\[ e^A B e^{-A} = B + \frac{1}{1!} [A, B] + \frac{1}{2!} [A, [A, B]] + \frac{1}{3!} [A, [A, [A, B]]] + \cdots, \]

show that

\[ \hat{S}^\dagger (\varepsilon) \hat{b} \hat{S}(\varepsilon) = \hat{b} \cosh r - \hat{b}^\dagger e^{2i\phi} \sinh r \]
\[ \hat{S}^\dagger (\varepsilon) \hat{b}^\dagger \hat{S}(\varepsilon) = \hat{b}^\dagger \cosh r - \hat{b} e^{-2i\phi} \sinh r \]

and

\[ \hat{S}^\dagger (\varepsilon) f(\hat{b}, \hat{b}^\dagger) \hat{S}(\varepsilon) = f(\hat{b} \cosh r - \hat{b}^\dagger e^{2i\phi} \sinh r , \hat{b}^\dagger \cosh r - \hat{b} e^{-2i\phi} \sinh r) , \]

(2)

where

\[ \hat{S}(\varepsilon) = \exp \left[ \frac{1}{2} (\varepsilon^* \hat{b}^2 - \varepsilon (\hat{b}^\dagger)^2) \right], \quad \varepsilon = re^{2i\phi}. \]

(b) Squeeze and Displacement operator (bonus)

The order of the \( \hat{D} \) and \( \hat{S} \) operators in the definition of a squeezed state in Eq. (36) (see lecture) does matter, since

\[ \hat{D}(\alpha) \hat{S}(\varepsilon) = \hat{S}(\varepsilon) \hat{D}(\gamma). \]

Express the parameter \( \gamma \) in terms of \( \alpha \) and \( \varepsilon \).

Hint: Use the transformation given in eq. (2).

(c) Properties of squeezed states (bonus)

Calculate the mean photon number

\[ \bar{n} = \langle \hat{n} \rangle = \langle \alpha, \varepsilon | \hat{n} | \alpha, \varepsilon \rangle = \langle \alpha, \varepsilon | \hat{b}^\dagger \hat{b} | \alpha, \varepsilon \rangle \]

and the variance of the photon probability distribution

\[ (\Delta n)^2 = \langle \alpha, \varepsilon | \hat{n}^2 | \alpha, \varepsilon \rangle - (\bar{n})^2 \]

for the squeezed state \( |\alpha, \varepsilon \rangle \).