1. Coherence Property of Light

Calculate the second-order correlation

\[ g^{(2)} (0) = \frac{\langle \hat{b}^\dagger \hat{b}^\dagger \hat{b} \hat{b} \rangle}{\langle \hat{b}^\dagger \hat{b} \rangle^2} \]

for the

(a) coherent state \( |\alpha\rangle \),
(b) Fock state \( |n\rangle \),
(c) Squeezed state \( |\alpha, r\rangle \).

2. Photon Statistics

Find the second-order correlation function

\[ g^{(2)} (0) = \frac{\langle \hat{b}^\dagger (t) \hat{b}^\dagger (t) \hat{b} (t) \hat{b} (t) \rangle}{\langle \hat{b}^\dagger (t) \hat{b} (t) \rangle^2} \]

of the light produced via parametric down conversion from

(a) Vacuum state

an initial vacuum state. Show that light exhibits bunching \( (g^{(2)} (0) > 1) \).

(b) Coherent state

an initial coherent state. Show that in this case the output will exhibit antibunching \( (g^{(2)} (0) < 1) \).
3. Single qubit systems

(a) Bloch vector of pure states
Show that all pure states of the single qubit density operator ($\rho_{\text{pure}}$) are represented by a Bloch vector ($\vec{s}$) with length one.

(b) Bloch vectors of orthogonal states
Which relation do the Bloch vectors $\vec{s}_1$ and $\vec{s}_2$ of two orthogonal single qubit density operators (i.e. $\text{tr}(\rho_1\rho_2) = 0$) have? Can orthogonal qubit density operators be mixed?

(Hint: use $\sigma_i\sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k$ and $\text{tr}(\sigma_i) = 0$)

4. Entanglement

(a) Schmidt decomposition
Find the Schmidt ranks of the two states

$$|\Psi_1\rangle = \frac{1}{2} (|00\rangle - i|01\rangle - i|10\rangle - |11\rangle),$$

$$|\Psi_2\rangle = \frac{1}{2} (|00\rangle + i|01\rangle + i|10\rangle + |11\rangle).$$

Are the states separable or entangled?

(b) Partial trace and entanglement
Apply the unitary transformation

$$U_{\text{cNOT}} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}$$

to the product state

$$\rho_{ab} = |\Psi\rangle\langle\Psi| = \frac{1}{2} \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1
\end{pmatrix},$$

with $|\Psi\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle)|1\rangle$, to get $\rho'_{ab} = U_{\text{cNOT}}\rho_{ab}U_{\text{cNOT}}^\dagger$. Calculate $\rho'_a = \text{tr}_b(\rho'_{ab})$ and describe the properties of that state. Is $\rho'_a$ separable or entangled?