1. *P* and *W*igner functions

(a) **Gaussian convolution**

Show that the Wigner function is a Gaussian convolution of the *P* function, i.e.

\[ W(\alpha) = \frac{2}{\pi} \int P(\beta) \exp \left[ -2|\beta - \alpha|^2 \right] d\beta^2 \]

(b) **Representation of the Wigner function**

Show that the Wigner function in suitable variables

\[ \alpha = \sqrt{\frac{m\omega}{2\hbar}} x + \frac{i}{\sqrt{2hm\omega}} p \]

can be represented as

\[ W(\alpha) = \frac{1}{\pi} \int \langle x - \frac{y}{2} | \hat{\rho} | x + \frac{y}{2} \rangle \exp \left( \frac{ipy}{\hbar} \right) dy \]

*Hint*: Show that the characteristic function \( \chi(\eta) \) of a state \( \hat{\rho} \) can be represented by

\[ \chi(\eta) = \int \int \langle x | \hat{\rho} | \tilde{x} \rangle \langle \tilde{x} | \exp \left( \eta \hat{b}^\dagger - \eta^* \hat{b} \right) | x \rangle dx d\tilde{x}. \]

Express \( \hat{b} \) and \( \hat{b}^\dagger \) in terms of position and momentum operators. Show that \( \exp \left[ -i \frac{p}{\hbar} a \right] |x\rangle = |x + a\rangle \).

(c) **Characteristic function of a Fock state**

Calculate the characteristic function of a Fock state \( |n\rangle \), i.e.

\[ \text{tr} \left[ |n\rangle \langle n| \exp \left( \eta \hat{b}^\dagger - \eta^* \hat{b} \right) \right] \]

(d) **Wigner function of a Fock state**

Evaluate the Wigner function for the Fock state \( |n = 1\rangle \) and show that it can be negative.
2. Angular momentum and the Jaynes-Cummings Model

(a) Angular momentum operators

Show that

\[
\begin{align*}
\hat{J}_x &= \frac{(\hat{b}^\dagger \hat{\sigma}_- + \hat{b} \hat{\sigma}_+)}{2\sqrt{\hat{M}}} \\
\hat{J}_y &= \frac{i(\hat{b}^\dagger \hat{\sigma}_- - \hat{b} \hat{\sigma}_+)}{2\sqrt{\hat{M}}} \\
\hat{J}_z &= \hat{\sigma}_z \\
\hat{\sigma}_z &= \frac{\hat{\sigma}_+ \hat{\sigma}_- - \hat{\sigma}_- \hat{\sigma}_+}{2} \\
\hat{M} &= \hat{b}^\dagger \hat{b} + \hat{\sigma}_+ \hat{\sigma}_-,
\end{align*}
\]

obey the algebra of the angular momentum operators, i.e. \([\hat{J}_x, \hat{J}_y] = i\hat{J}_z, \quad [\hat{J}_y, \hat{J}_z] = i\hat{J}_x\) and \([\hat{J}_z, \hat{J}_x] = i\hat{J}_y\), and show that \(\hat{M}\) is a conserved quantity.

(Hint: Remember the relations \([b, b^\dagger] = 1\) and \(\{\sigma_+, \sigma_-\} = 1\).)

(b) Spin in an effective magnetic field

Show that the Jaynes-Cummings model Hamiltonian

\[
\hat{H} = \hbar \omega_0 \hat{\sigma}_z + \hbar \omega \hat{b}^\dagger \hat{b} + \hbar g (\hat{b}^\dagger \hat{\sigma}_- + \hat{b} \hat{\sigma}_+) \tag{2}
\]

describes a spin in an effective magnetic field, i.e. find a \(\mathbf{B}\) such that

\[
\hat{H} = \hat{\mathbf{J}} \cdot \mathbf{B} + c, \tag{3}
\]

where the components of the vector \(\hat{\mathbf{J}}\) are given by equation (1) and \(c\) is an arbitrary energy shift.

(c) Spectrum of the Hamiltonian

Find the spectrum of the Hamiltonian \(\hat{H}\) using either Eq. (2) or (3).

Note, consistent with the lecture we use: \(\hat{\sigma}_z = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\).