1. **Gauge transformation**

The interaction of the electromagnetic field with atoms can be described by the following Schrödinger equation

\[ i \hbar \frac{\partial}{\partial t} \Psi (\vec{r}, t) = \frac{1}{2m} \left( -i \hbar \vec{\nabla} - \frac{e}{c} \vec{A} \right)^2 \Psi (\vec{r}, t) + V (\vec{r}) \Psi (\vec{r}, t) \]

where \( \vec{A} \) is the vector potential of the electromagnetic field. \( V (\vec{r}) \) is the potential energy of the electron.

(a) **Coulomb gauge**

Show that this equation may be written as

\[ i \hbar \frac{\partial}{\partial t} \Psi (\vec{r}, t) = -\frac{\hbar^2}{2m} \vec{\nabla}^2 \Psi (\vec{r}, t) + \frac{e}{mc} \vec{A} \cdot \vec{\nabla} \Psi (\vec{r}, t) + \frac{e^2}{2mc^2} \vec{A}^2 \Psi (\vec{r}, t) + V (\vec{r}) \Psi (\vec{r}, t) \]  \( (1) \)

in the Coulomb gauge, i.e. \( \vec{\nabla} \cdot \vec{A} = 0 \).

(b) **Gauge transformation**

Find a gauge transformation \( \Phi (\vec{r}, t) = \Omega \Psi (\vec{r}, t) \) to eliminate the term \( \vec{A} \cdot \vec{\nabla} \Psi (\vec{r}, t) \) from eq. (1).

(c) **Schrödinger equation**

Write the Schrödinger equation for the new wave function \( \Phi (\vec{r}, t) \).

2. **Density matrix**

(a) **Von Neumann equation**

Show that the density matrix

\[ \rho (t) = \langle \Psi (t) \rangle \langle \Psi (t) \rangle \]
obeys the equation
\[
i\hbar \frac{d\rho(t)}{dt} = [\hat{H}, \rho(t)],
\] (2)

where $\hat{H}$ is the Hamiltonian of the system.

(b) **Trace preservation**
Use (2) to show that
\[
\frac{d}{dt} \text{tr}[\rho(t)] = 0.
\]

(c) **Heisenberg equation**
Derive the Heisenberg equation for an operator $\hat{A}$, i.e.
\[
i\hbar \frac{d\hat{A}(t)}{dt} = [\hat{A}(t), \hat{H}],
\] (3)
from eq. (2) by starting with $\frac{d}{dt} \langle \hat{A} \rangle = \text{tr} \left( \frac{d}{dt} \hat{A} \right)$ in the Schrödinger picture.

(d) **Schrödinger vs. Heisenberg picture**
What is the difference between the Schrödinger (eq. 2) and the Heisenberg picture (eq. 3)?

3. **"Displacement" Operator**
By using $e^{\hat{A} + \hat{B}} = e^{\hat{A}} e^{\hat{B}} e^{-\frac{1}{2}[\hat{A},\hat{B}]}$, which holds if both $\hat{A}$ and $\hat{B}$ commute with $[\hat{A},\hat{B}]$, show that
\[
\hat{D}^\dagger (\alpha) \hat{b} \hat{D} (\alpha) = \hat{b} + \alpha,
\]
\[
\hat{D}^\dagger (\alpha) \hat{b}^\dagger \hat{D} (\alpha) = \hat{b}^\dagger + \alpha^*,
\]
\[
\hat{D} (\alpha + \beta) = \hat{D} (\alpha) \hat{D} (\beta) e^{-i\text{Im}(\alpha\beta^*)},
\]
where
\[
\hat{D} (\alpha) = \exp \left( \alpha \hat{b}^\dagger - \alpha^* \hat{b} \right).
\]