1. Entanglement

Consider the density operator

\[ \rho_{ABC} = \frac{1}{N} \left[ 2|GHZ\rangle\langle GHZ| + \sum_{x=1}^{3} a_x |k_x\rangle\langle k_x| + \frac{1}{a_x} |\bar{k}_x\rangle\langle \bar{k}_x| \right] \]

with \( k \in \{k_1 = 001, k_2 = 010, k_3 = 100\} \), \( \bar{k} \) being the binary representation and which is given by inverting all digits of \( k \), \( |GHZ\rangle = \frac{1}{\sqrt{2}} (|000\rangle + |111\rangle) \) and \( \prod_{x=1}^{3} a_x \neq 1 \)

(a) Calculate the normalization \( N \) and write \( \rho_{ABC} \) as a matrix in the computational basis. What are the restrictions on \( a_x \) such that \( \rho_{ABC} \) is positive?

(b) Show that the states have a positive partial transpose with respect to every bipartite split. (Hint: Make use of the block structure of \( \rho_{ABC}^{\Gamma_I} \), where \( \Gamma_I \) denotes the partial transposition with respect to subsystems \( I \in \{A, B, C\} \)).

(c) Show that the states are separable with respect to every bipartite split. (Hint: Again use the block structure and that in 2 \( \times \) 2-dimensions the PPT criterion is necessary and sufficient for separability).

(d) Show that the states \( \rho_{ABC} \) are entangled by using the range criterion. You may use the following procedure:

Show that there exists no product vector \( |\phi\rangle = \bigotimes_{m=1}^{3} |\phi_m\rangle \) with the conditions \( |\phi\rangle \in \text{range}(\rho_{ABC}) \) (C1) and \( |\phi\rangle^* \in \text{range}(\rho_{ABC}^{\Gamma_I}) \) (C2).

(i) Obtain the kernels of \( \rho_{ABC} \) and \( \rho_{ABC}^{\Gamma_I} \).

(ii) Reformulate the conditions C1 and C2 in terms of the kernels.

(iii) It can be shown that \( |\phi_m\rangle = \cos \theta_m |0\rangle + \sin \theta_m |1\rangle \) with \( \theta \in [0, \frac{\pi}{4}] \) is in this case a general representation of \( |\phi_m\rangle \). Combine the conditions (C1, C2) in order to get a contradiction to \( \prod_{x=1}^{3} a_x \neq 1 \).
2. State tomography

Let \( \rho \) be an unknown qubit state and \( E_1 = \frac{1}{3}|0\rangle\langle 0|, \ E_2 = \frac{1}{3}|+\rangle\langle +|, \ E_3 = \frac{1}{3}|i+\rangle\langle i+| \) and \( E_4 = \mathbf{1} - (E_1 + E_2 + E_3) \) being a POVM, where \( |+\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle), |i+\rangle = \frac{1}{\sqrt{2}} (|0\rangle + i|1\rangle) \) and \( \{ |i\rangle \} \) denotes the computational basis.

(a) Formulate the set of linear equations of \( p_l = \text{tr}(\rho E_l) \) by using \( E_l = \sum_{i,j}^{1,1} c_{ij}^{[l]} |i\rangle\langle j| \) and \( \rho = \sum_{i,j}^{1,1} r_{ij} |i\rangle\langle j| \) with \( \langle i| j \rangle = \delta_{ij} \) for the unknown variables \( r_{ij} \).

(b) The application of \( n \) identical and independent POVMs \( \{ E_l \}_{l=1}^{4} \) on \( n \) copies of \( \rho \) leads to the empirical probabilities \( p_{nl} \approx \text{tr}(\rho E_l) \). For \( n = 650 \) this gives \( p_{n1} = \frac{159}{650}, p_{n2} = \frac{123}{650}, p_{n3} = \frac{140}{650} \) and \( p_{n4} = \frac{228}{650} \). Reconstruct the state \( \rho_n \) by using the empirical probabilities.

(c) Consider the set of states \( \Gamma_\zeta \), where the distance (1-norm) between any \( \rho \in \Gamma_\zeta \) and \( \rho_n \) is at most \( \zeta = 0.3 \). What is at maximum the probability that \( \rho \) does not belong to the set \( \Gamma_\zeta \)?

(d) What is the form of the set of states \( \Gamma_\zeta \) in the Bloch vector representation? Calculate the relative volume of the set \( \Gamma_\zeta \) with respect to the volume of the whole Bloch sphere. (Hint: Note that every \( 2 \times 2 \)-density operator \( m \) can be written as \( m = \frac{1}{2} (1 + \vec{\sigma}) \) with \( \vec{\sigma} = \sigma_x e_x + \sigma_y e_y + \sigma_z e_z \) and \( \sigma_i \) Pauli matrices).