1. **Angular momentum and the Jaynes-Cummings Model**

(a) **Angular momentum operators**

Show that

\[
\hat{J}_x = \frac{(\hat{b}^\dagger \hat{\sigma}_- + \hat{b} \hat{\sigma}_+)}{2\sqrt{\hat{M}}}
\]
\[
\hat{J}_y = \frac{i(\hat{b}^\dagger \hat{\sigma}_- - \hat{b} \hat{\sigma}_+)}{2\sqrt{\hat{M}}}
\]
\[
\hat{J}_z = \hat{\sigma}_z
\]
\[
\hat{\sigma}_z = \frac{\hat{\sigma}_+ \hat{\sigma}_- - \hat{\sigma}_- \hat{\sigma}_+}{2}
\]
\[
\hat{M} = \hat{b}^\dagger \hat{b} + \hat{\sigma}_+ \hat{\sigma}_-
\]

obey the algebra of the angular momentum operators, i.e. \([\hat{J}_x, \hat{J}_y] = i\hat{J}_z\), \([\hat{J}_y, \hat{J}_z] = i\hat{J}_x\) and \([\hat{J}_z, \hat{J}_x] = i\hat{J}_y\), and show that \(\hat{M}\) is a conserved quantity.

*(Hint: Remember the relations \([b, b^\dagger] = 1\) and \(\{\sigma_+, \sigma_-\} = 1\).*

(b) **Spin in an effective magnetic field**

Show that the Jaynes-Cummings model Hamiltonian

\[
\hat{H} = \hbar \omega_0 \hat{\sigma}_z + \hbar \omega \hat{b}^\dagger \hat{b} + \hbar g (\hat{b}^\dagger \hat{\sigma}_- + \hat{b} \hat{\sigma}_+)
\]

(2)

describes a spin in an effective magnetic field, i.e. find a \(\mathbf{B}\) such that

\[
\hat{H} = \hat{\mathbf{J}} \cdot \mathbf{B} + c,
\]

(3)

where the components of the vector \(\hat{\mathbf{J}}\) are given by equation (1) and \(c\) is an arbitrary energy shift.

(c) **Spectrum of the Hamiltonian**

Find the spectrum of the Hamiltonian in equation (2) by using the form given in equation (3).

Note, consistent with the lecture we use:

\[
\hat{\sigma}_z = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]