1. Partial Transpose

Let $\rho$ be a density operator. Use the partial transpose criterion, i.e.

$$\rho^{TA} \not\succeq 0 \Rightarrow \rho \text{ is entangled},$$

to show for which $p$ the Werner state,

$$\rho_w = p|\Phi^+\rangle\langle\Phi^+| + \frac{(1-p)}{4} \mathbb{1} \quad \text{with} \quad |\Phi^+\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle) \quad (1)$$

and $\mathbb{1}$ being the identity, is entangled.

2. Reduction criterion

(a) Prove the reduction criterion, which reads:

$$\rho \text{ separable} \Rightarrow \rho_A \otimes \mathbb{1} - \rho \succeq 0 \quad \text{and} \quad \mathbb{1} \otimes \rho_B - \rho \succeq 0,$$

where $\rho_A := \text{tr}_B \rho$ and $\rho_B := \text{tr}_A \rho$.

Hint: First show that the map $\Lambda : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$, $\Lambda(\rho) := (\text{tr}\rho)\mathbb{1} - \rho$, $\rho \in \mathcal{B}(\mathcal{H})$, maps positive\(^1\) operators to positive operators. Then use the fact that applying this map to one subsystem of a separable state has to lead to a positive matrix.

(b) Apply the reduction criterion to the Werner state (eq. 1) and find the corresponding threshold for $p$.

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\(^1\)By saying “$\rho$ is positive” we mean “positive semi-definite”, i.e. $\langle \psi | \rho | \psi \rangle \geq 0, \forall |\psi\rangle$. 
3. **Majorisation criterion**

Apply the majorisation criterion to the Werner state (eq. 1) and find the corresponding threshold for $p$.

4. **Witness operators**

(a) Calculate the matrix representation of the witness operator

$$W_w = \left( |\Psi_{\text{neg}}\rangle\langle \Psi_{\text{neg}}| \right)^{T_A}$$

with $\rho_w^{T_A} |\Psi_{\text{neg}}\rangle = -|\lambda_{\text{min}}\rangle|\Psi_{\text{neg}}\rangle$

in the computational basis using the Werner state of eq. 1.

(b) For which $p$ is $\text{tr}(W_w \rho_w) < 0$ fulfilled? Does this witness detect all entangled $\rho_w$?

(c) Given $W = x \mathbb{1} - |\Psi\rangle\langle \Psi|$ where $x = \max_{|\phi\rangle \in S} |\langle \phi | \Psi\rangle|^2$, $|\Psi\rangle$ entangled, and $S$ being the set of separable states. Show that $W$ is an entanglement witness.

(d) Write $W_w$ as $W_w = x \mathbb{1} - |\Psi_{\text{ent}}\rangle\langle \Psi_{\text{ent}}|$, i.e. calculate $x$ and the state $|\Psi_{\text{ent}}\rangle$. 