Theoretical Quantum Optics and Quantum Information (SS 07)
Exercise VIII (13.06.07)

1. Single qubit systems

(a) Bloch vector of pure states
   Show that all pure states of the single qubit density operator ($\rho_{\text{pure}}$) are represented by a Bloch vector ($\vec{s}$) with length one.

(b) Bloch vectors of orthogonal states
   Which relation do the Bloch vectors $\vec{s}_1$ and $\vec{s}_2$ of two orthogonal single qubit density operators (i.e. $\text{tr}(\rho_1 \rho_2) = 0$) have? (Hint: use $\sigma_i \sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k$ and $\text{tr}(\sigma_i = 0)$)
   Can orthogonal density operators be mixed?

(c) Mixed states
   Prove that the decomposition of a mixed state density matrix into a weighted sum of projectors is not unique. (Give an example of two different weighted sums of projectors onto pure states, leading to the same mixed density operator.)

2. Composite qubit systems

(a) Tensor product I
   Construct the state vector of the combined system $|ab\rangle$ using the single qubit state vectors $|a\rangle = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $|b\rangle = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$.

(b) Tensor product II
   Construct the density matrices $\rho_a$ and $\rho_b$, using the state $|a\rangle$ and $|b\rangle$, respectively. Calculate the density matrix of the combined system ($\rho_{ab}$) by building the tensor product $\rho_{ab} = \rho_a \otimes \rho_b$. Verify the result by comparing it with $|ab\rangle\langle ab|$ from (2a).

(c) Partial trace
   Verify that $\rho_b = \text{tr}_a(\rho_{ab})$ by using the expressions from (2b).
(d) **Unitary Transformation**

Calculate \( \rho'_a = U_a \rho_a U_a^\dagger \) and \( \rho'_b = U_b \rho_b U_b^\dagger \) for

\[
U_a = \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix}, \quad U_b = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}.
\]

(e) **Unitary Local Transformation and Partial Traces**

Verify that \( \rho'_{ab} = U_{ab} \rho_{ab} U_{ab}^\dagger \) where \( \rho'_{ab} = \rho'_a \otimes \rho'_b \) and \( U_{ab} = U_a \otimes U_b \). Check whether \( \text{tr}_b(\rho'_{ab}) = \rho'_a \).

3. **Entanglement**

(a) **Schmidt Decomposition**

Find the Schmidt ranks of the two states

\[
|\Psi_1\rangle = \frac{1}{2} (i |00\rangle + |01\rangle - |10\rangle + i |11\rangle),
\]

\[
|\Psi_2\rangle = \frac{1}{2} (i |00\rangle + |01\rangle + |10\rangle + i |11\rangle).
\]

Are the states separable or entangled?

(b) **Partial Trace and Entanglement**

Apply the unitary transformation

\[
U_{\text{cNOT}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}
\]

to the product state

\[
\rho_{ab} = |\Psi\rangle\langle \Psi| = \frac{1}{2} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
\]

with \( |\Psi\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) |0\rangle \), to get \( \rho'_{ab} = U_{\text{cNOT}} \rho_{ab} U_{\text{cNOT}}^\dagger \) (Note: The unitary transformation represented by \( U_{\text{cNOT}} \) cannot be written as product of local rotations, i.e. \( U_{\text{cNOT}} \neq U_a \otimes U_b \)).

Calculate \( \rho'_a = \text{tr}_b(\rho'_{ab}) \) and describe the properties of that state. Is \( \rho'_{ab} \) separable or entangled?