Quantum Optics and Information (SS 06)
Exercise XIII (05.07.06)

1. Kraus operator representation

A qubit in the depolarizing channel is subjected to the following error model:
With probability $1 - p$ nothing happens to the qubit, and with probability $p$ one of the following errors occur with equal probability:

- Bit flip: $|0\rangle \rightarrow |1\rangle$, $|1\rangle \rightarrow |0\rangle$, i.e. $|\psi\rangle \rightarrow \sigma_x |\psi\rangle$.
- Phase flip: $|0\rangle \rightarrow |0\rangle$, $|1\rangle \rightarrow -|1\rangle$, i.e. $|\psi\rangle \rightarrow \sigma_z |\psi\rangle$.
- Bit and phase flip: $|0\rangle \rightarrow i|1\rangle$, $|1\rangle \rightarrow -i|0\rangle$, i.e. $|\psi\rangle \rightarrow \sigma_y |\psi\rangle$.

Write the effect of such a channel on a general (mixed) qubit state $\rho$ in its Kraus operator representation $\rho' = \sum_i A_i \rho A_i^\dagger$ and give an explicit expression for the Kraus operators.

2. Local filtering

Consider a pure two qubit quantum state $|\Psi\rangle = c_1 |00\rangle + c_2 |11\rangle$ with real coefficients $c_1 \geq c_2 > 0$ and $c_1^2 + c_2^2 = 1$. Apply positive local operators $F_A = a_1 |0\rangle \langle 0| + a_2 |1\rangle \langle 1|$ and $F_B = b_1 |0\rangle \langle 0| + b_2 |1\rangle \langle 1|$ with $\mathbb{I} - F_A \otimes F_B \geq 0$ to the state $|\Psi\rangle$, such that $F_A \otimes F_B |\Psi\rangle = \sqrt{p} (|00\rangle + |11\rangle)$. Calculate the coefficients $a_1, a_2, b_1, b_2$ and the optimal success probability $p$ of such a process.
3. **Unextendible Product Bases (UPB)**

The set of states $S = \{ |0, 1, +\rangle, |1, +, 0\rangle, |+, 0, 1\rangle, |-, -, -\rangle \}$, with $|\pm\rangle = \frac{1}{\sqrt{2}} (|0\rangle \pm |1\rangle)$, form a three qubit UPB, i.e. there exists no further product state which is orthogonal to all states in $S$.

(a) Show that the state $\rho_{BE} := \frac{1}{4} (1 - \sum_{s \in S} |s\rangle\langle s|)$ is entangled

*(Hints: Use (and prove) the following theorem: A state $\rho$ is separable iff there exists a set of product vectors that spans the range of $\rho$, and their partially complex conjugates span the range of $\rho^{T_A}$.)*

Remember: The complete Hilbert space is spanned by the UPB ($\mathcal{H}_{UPB}$) and the orthocomplement ($\mathcal{H}^\perp$), i.e. $\mathcal{H} = \mathcal{H}_{UPB} \oplus \mathcal{H}^\perp$.

(b) Show that $\rho_{BE}$ has a positive partial transposition with respect to any bipartite split. What does that mean with respect to distillability?

(c) Is there entanglement between any bipartite split?

4. **Concurrence**

For two qubit systems the entanglement of formation $E_F(\rho)$ can be calculated explicitly with “Wootters” formula,

$$E_F(\rho) = H \left( \frac{1}{2} + \frac{1}{2} \sqrt{1 - C^2} \right),$$

using the Shannon entropy $H(x) = -x \log x - (1 - x) \log (1 - x)$ and the concurrence $C$. The concurrence is given by

$$C(\rho) = \max\{0, \sqrt{\xi_1} - \sqrt{\xi_2} - \sqrt{\xi_3} - \sqrt{\xi_4} \}$$

where $\xi_i$ are the ordered real eigenvalues, i.e. $\xi_1 \geq \xi_2 \geq \xi_3 \geq \xi_4$, of $\rho ((\sigma_y \otimes \sigma_y) \rho^* (\sigma_y \otimes \sigma_y))$.

Calculate the entanglement of formation for a two qubit Werner state,

$$\rho_w = p|\Phi^+\rangle\langle\Phi^+| + \frac{(1 - p)}{4} \mathbb{1} \quad \text{with} \quad |\Phi^+\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle),$$

as a function of the parameter $p$. 