Quantum Optics and Information (SS 06)

Exercise XIII (05.07.06)

1. KRAUS OPERATOR REPRESENTATION

   A qubit in the depolarizing channel is subjected to the following error model: With probability $1-p$ nothing happens to the qubit, and with probability p one of the following errors occur with equal probability:

   - Bit flip: $|0\rangle \rightarrow |1\rangle$, $|1\rangle \rightarrow |0\rangle$, i.e. $|\psi\rangle \rightarrow \sigma_x |\psi\rangle$.
   - Phase flip: $|0\rangle \rightarrow |0\rangle$, $|1\rangle \rightarrow -|1\rangle$, i.e. $|\psi\rangle \rightarrow \sigma_z |\psi\rangle$.
   - Bit and phase flip: $|0\rangle \rightarrow i|1\rangle$, $|1\rangle \rightarrow -i|0\rangle$, i.e. $|\psi\rangle \rightarrow \sigma_y |\psi\rangle$.

   Write the effect of such a channel on a general (mixed) qubit state ρ in its Kraus operator representation $\rho' = \sum_i A_i \rho A_i^\dagger$ and give an explicit expression for the Kraus operators.

2. LOCAL FILTERING

   Consider a pure two qubit quantum state $|\Psi\rangle = c_1 |00\rangle + c_2 |11\rangle$ with real coefficients $c_1 \geq c_2 > 0$ and $c_1^2 + c_2^2 = 1$. Apply positive local operators $F_A = a_1 \big|0\big\rangle\big\langle 0\big| + a_2 \big|1\big\rangle\big\langle 1\big|$ and $F_B = b_1 \big|0\big\rangle\big\langle 0\big| + b_2 \big|1\big\rangle\big\langle 1\big|$ with $\mathbb{1} - F_A \otimes F_B \geq 0$ to the state $|\Psi\rangle$, such that $F_A \otimes F_B |\Psi\rangle = \sqrt{\frac{p}{2}} (|00\rangle + |11\rangle)$. Calculate the coefficients a_1, a_2, b_1, b_2 and the optimal success probability p of such a process.

3. Unextendible product bases (UPB)

The set of states $S = \{|0,1,+\rangle, |1,+,0\rangle, |+,0,1\rangle, |-,-,-\rangle\}$, with $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$, form a three qubit UPB, i.e. there exists no further product state which is orthogonal to all states in S.

(a) Show that the state $\rho_{\mathrm{BE}} := \frac{1}{4}\left(\mathbb{1} - \sum_{s\in S} |s\rangle\langle s|\right)$ is entangled

(*Hints:* Use (and prove) the following theorem: A state ρ is separable iff there exists a set of product vectors that spans the range of ρ, and their partially complex conjugates span the range of ρ^{T_A}.

Remember: The complete Hilbert space is spanned by the UPB ($\mathcal{H}_{\mathrm{UPB}}$) and the orthocomplement ($\mathcal{H}^{\perp}$), i.e. $\mathcal{H} = \mathcal{H}_{\mathrm{UPB}} \oplus \mathcal{H}^{\perp}$.).

(b) Show that ρ_{BE} has a positive partial transposition with respect to any bipartite split. What does that mean with respect to distillability?

(c) Is there entanglement between any bipartite split?

4. Concurrence

For two qubit systems the entanglement of formation $E_F(\rho)$ can be calculated explicitly with "Wootters" formula,

$$E_F(\rho) = H\left(\frac{1}{2} + \frac{1}{2}\sqrt{1 - C^2}\right),$$

using the Shannon entropy $H(x) = -x\log x - (1-x)\log(1-x)$ and the concurrence C. The concurrence is given by

$$C(\rho) = \max\{0, \sqrt{\xi_1} - \sqrt{\xi_2} - \sqrt{\xi_3} - \sqrt{\xi_4}\}$$

where ξ_i are the ordered real eigenvalues, i.e. $\xi_1 \geq \xi_2 \geq \xi_3 \geq \xi_4$, of $\rho\left((\sigma_y \otimes \sigma_y)\,\rho^*\,(\sigma_y \otimes \sigma_y)\right)$.

Calculate the entanglement of formation for a two qubit Werner state,

$$\rho_{\mathrm{w}} = p|\Phi^+\rangle\langle\Phi^+| + \frac{(1-p)}{4}\mathbb{1} \quad \text{with} \quad |\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle),$$

as a function of the parameter p.