Quantum Optics and Information (SS 06)

Exercise IX (07.06.06)

1. **Von Neumann entropy of pure states**

   Calculate the von Neumann entropy of the reduced density operator \( S(\rho_a) \) of qubit \( a \) \( (\rho_a = \text{tr}_b(\rho) = \text{tr}_b(\rho_a) = \text{tr}_b(\rho_{\Psi}) = \frac{1}{\sqrt{2}} (|00\rangle + \cos \phi |10\rangle + \sin \phi |11\rangle) \) of the two-qubit states, (i) \( |\Psi_1\rangle = \alpha |00\rangle + \beta |11\rangle \) with \( |\alpha|^2 + |\beta|^2 = 1 \) and (ii) \( |\Psi_{ii}\rangle = \frac{1}{\sqrt{2}} (|00\rangle) \). Draw a graph showing the von Neumann entropy depending on (i) \( |\alpha|^2 \) and (ii) \( \cos \phi \).

   What are the properties of \( |\Psi_1\rangle \) and \( |\Psi_{ii}\rangle \)?

2. **Measurements**

   (a) **Projectors**

   Let \( |u\rangle \) and \( |v\rangle \) be normalized vectors. Show that \( |u\rangle \langle u| \) and \( |v\rangle \langle v| \) are projectors. Moreover, show that \( |u\rangle \langle u| + |v\rangle \langle v| \) is a projector iff \( \langle u|v\rangle = 0 \).

   Generalize this result to an arbitrary number of vectors.

   (b) **von Neumann measurements**

   The measurement performed by a Stern-Gerlach apparatus oriented in z-direction is a set of orthogonal projectors, \( P_1 = |0\rangle \langle 0| \) and \( P_2 = |1\rangle \langle 1| \).

   Show that this apparatus cannot distinguish between \( |\psi_1\rangle := \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle) \) and \( \rho_2 := \frac{1}{2} |0\rangle \langle 0| + \frac{1}{2} |1\rangle \langle 1| \). Devise a measurement achieving that.

3. **Entanglement and decoherence**

   (a) **Density operator of the singlet**

   Given the singlet state \( |\psi^-\rangle = \frac{1}{\sqrt{2}} (|01\rangle - |10\rangle) \).

   i) Write \( |\psi^-\rangle \) in another basis \( \{ |\bar{0}\rangle, |\bar{1}\rangle \} \), defined by \( |\bar{0}\rangle = \alpha |0\rangle + \beta |1\rangle \) and \( |\bar{1}\rangle = -\beta^* |0\rangle + \alpha^* |1\rangle \), with \( |\alpha|^2 + |\beta|^2 = 1 \).

   ii) Show that \( |\psi^-\rangle \langle \psi^-| \) can be written as \( \rho = \frac{1}{4} (1 \otimes 1 - \sum_i \sigma_i \otimes \sigma_i) \) with \( \sigma_i \) being the Pauli matrices.
(b) The depolarizing channel

A qubit in the depolarizing channel is subjected to the following error model:
With probability $1 - p$ nothing happens to the qubit, and with probability $p$ one of the following errors occur with equal probability:

- **Bit flip:** $|0\rangle \rightarrow |1\rangle$, $|1\rangle \rightarrow |0\rangle$, i.e. $|\psi\rangle \rightarrow \sigma_x |\psi\rangle$.
- **Phase flip:** $|0\rangle \rightarrow |0\rangle$, $|1\rangle \rightarrow -|1\rangle$, i.e. $|\psi\rangle \rightarrow \sigma_z |\psi\rangle$.
- **Bit and phase flip:** $|0\rangle \rightarrow i|1\rangle$, $|1\rangle \rightarrow -i|0\rangle$, i.e. $|\psi\rangle \rightarrow \sigma_y |\psi\rangle$.

This channel is described by the following unitary evolution in the Hilbert space of the qubit plus an environment:

$$
|\psi\rangle_A \otimes |0\rangle_E \rightarrow \sqrt{1 - p} |\psi\rangle_A \otimes |0\rangle_E + \sqrt{p/3} \sum_i \sigma_i |\psi\rangle_A \otimes |i\rangle_E .
$$

Compute the Bloch vector representation of a general pure state after this evolution.

4. Quantum teleportation

(a) Entanglement swapping

In the lecture the entanglement swapping protocol was presented. Show that a Bell measurement on particles $Z$ and $A$ in the state

$$
|\psi^{\text{total}}\rangle = |\psi\rangle_{YZ} \otimes |\psi^-\rangle_{AB} = [\alpha |01\rangle + \beta |10\rangle]_{YZ} \otimes \frac{1}{\sqrt{2}} ([01] - |10\rangle)]_{AB} ,
$$

followed by an appropriate rotation of the particles $B$, results in the desired state $|\psi\rangle_{BY}$, up to a global phase. Which rotation of $B$ is needed for which outcome of the Bell measurement?