Quantum Optics and Information (SS 06)

Exercise IX (07.06.06)

1. *Von Neumann entropy of pure states*

   Calculate the von Neumann entropy of the reduced density operator ($S(\rho_a)$) of qubit a ($\rho_a = \text{tr}_b(\rho) = \text{tr}_b\big(|\Psi\rangle\langle\Psi|\big)$) of the two-qubit states, (i) $|\Psi_{\text{i}}\rangle = \alpha\,|00\rangle + \beta\,|11\rangle$ with $|\alpha|^2 + |\beta|^2 = 1$ and (ii) $|\Psi_{\text{ii}}\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + \cos\phi\,|10\rangle + \sin\phi\,|11\rangle\right)$. Draw a graph showing the von Neumann entropy depending on (i) $|\alpha|^2$ and (ii) $\cos\phi$. What are the properties of $|\Psi_{\text{i}}\rangle$ and $|\Psi_{\text{ii}}\rangle$?

2. *Measurements*

   (a) Projectors

   Let $|u\rangle$ and $|v\rangle$ be normalized vectors. Show that $|u\rangle\langle u|$ and $|v\rangle\langle v|$ are projectors. Moreover, show that $|u\rangle\langle u| + |v\rangle\langle v|$ is a projector iff $\langle u\,|v\rangle = 0$. Generalize this result to an arbitrary number of vectors.

   (b) von Neumann measurements

   The measurement performed by a Stern-Gerlach apparatus oriented in z-direction is a set of orthogonal projectors, $P_1 = |0\rangle\langle 0|$ and $P_2 = |1\rangle\langle 1|$. Show that this apparatus cannot distinguish between $|\psi_1\rangle := \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$ and $\rho_2 := \frac{1}{2}\,|0\rangle\langle 0| + \frac{1}{2}\,|1\rangle\langle 1|$. Devise a measurement achieving that.

3. *Entanglement and decoherence*

   (a) Density operator of the singlet

   Given the singlet state

   $$|\psi^-\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right).$$

   i) Write $|\psi^-\rangle$ in another basis $\{|\bar{0}\rangle, |\bar{1}\rangle\}$, defined by $|\bar{0}\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$ and $|\bar{1}\rangle = -\beta^*\,|0\rangle + \alpha^*\,|1\rangle$, with $|\alpha|^2 + |\beta|^2 = 1$.

   ii) Show that $|\psi^-\rangle\langle\psi^-|$ can be written as $\rho = \frac{1}{4}\left(\mathbf{1}\otimes\mathbf{1} - \sum_i \sigma_i\otimes\sigma_i\right)$ with σ_i being the Pauli matrices.

(b) The depolarizing channel

A qubit in the depolarizing channel is subjected to the following error model: With probability $1-p$ nothing happens to the qubit, and with probability p one of the following errors occur with equal probability:

- Bit flip: $|0\rangle \to |1\rangle$, $|1\rangle \to |0\rangle$, i.e. $|\psi\rangle \to \sigma_x |\psi\rangle$.
- Phase flip: $|0\rangle \to |0\rangle$, $|1\rangle \to -|1\rangle$, i.e. $|\psi\rangle \to \sigma_z |\psi\rangle$.
- Bit and phase flip: $|0\rangle \to i|1\rangle$, $|1\rangle \to -i|0\rangle$, i.e. $|\psi\rangle \to \sigma_y |\psi\rangle$.

This channel is described by the following unitary evolution in the Hilbert space of the qubit plus an environment:

$$|\psi\rangle_{\mathrm{A}} \otimes |0\rangle_{\mathrm{E}} \to \sqrt{1-p}\, |\psi\rangle_{\mathrm{A}} \otimes |0\rangle_{\mathrm{E}} + \sqrt{\frac{p}{3}} \sum_i \sigma_i |\psi\rangle_{\mathrm{A}} \otimes |i\rangle_{\mathrm{E}} \,.$$

Compute the Bloch vector representation of a general pure state after this evolution.

4. *Quantum teleportation*

(a) Entanglement swapping

In the lecture the entanglement swapping protocol was presented. Show that a Bell measurement on particles Z and A in the state

$$\begin{aligned} \left|\psi^{\mathrm{total}}\right\rangle &= |\psi\rangle_{\mathrm{YZ}} \otimes \left|\psi^{-}\right\rangle_{\mathrm{AB}} \\ &= [\alpha\,|01\rangle + \beta\,|10\rangle]_{\mathrm{YZ}} \otimes \frac{1}{\sqrt{2}} [|01\rangle - |10\rangle]_{\mathrm{AB}} \,, \end{aligned}$$

followed by an appropriate rotation of the particles B, results in the desired state $|\psi\rangle_{\mathrm{BY}}$, up to a global phase. Which rotation of B is needed for which outcome of the Bell measurement?