I. JAYNES-CUMMINGS HAMILTONIAN

1. A two-state atom interacts on resonance with a single mode field. The atom-field interaction is described by the Jaynes-Cummings Hamiltonian

\[ H = \Omega \left( \hat{a}^{\dagger} \sigma_- + \hat{a} \sigma_+ \right), \]  

where

\[ [\hat{a}, \hat{a}^{\dagger}] = 1, \]
\[ \{\sigma_-, \sigma_+\} = 1. \]

Explain the physical meaning of all the terms.

2. Find the spectrum of the Hamiltonian (1),

Hint: Square the Hamiltonian.

3. Suppose that the atom is initially in the excited state \( |2\rangle \) and the field has \( n \) photons. Show that joint state of the atom + field system at some time \( t \) is given

\[ |\Psi(t)\rangle = \cos[\alpha(n)t] |2\rangle \otimes |n\rangle + i \sin[\alpha(n)t] |1\rangle \otimes |n+1\rangle. \]

where \( |1\rangle \) is the ground state of two-state atom.

Find the expression for \( \alpha(n) \).

II. ANGULAR MOMENTUM AND JAYNES-CUMMINGS MODEL

1. Show that

\[ \hat{J}_x = \left( \hat{a}^{\dagger} \sigma_- + \hat{a} \sigma_+ \right) / 2 \sqrt{\hat{M}} \]
\[ \hat{J}_y = i \left( \hat{a}^{\dagger} \sigma_- - \hat{a} \sigma_+ \right) / 2 \sqrt{\hat{M}} \]
\[ \hat{J}_z = \sigma_z / 2, \]
\[ \hat{\sigma}_z = \sigma_+ \sigma_- - \sigma_- \sigma_+ \]
\[ \hat{M} = \hat{a}^{\dagger} \hat{a} + \sigma_+ \sigma_- \]

obey the algebra of the angular momentum operators, i.e.

\[ [\hat{J}_x, \hat{J}_y] = i \hat{J}_z \]
\[ [\hat{J}_y, \hat{J}_z] = i \hat{J}_x \]
\[ [\hat{J}_z, \hat{J}_x] = i \hat{J}_y \]

2. Show that the Jaynes-Cummings model

\[ H = \frac{\hbar \omega_0}{2} \hat{\sigma}_z + \hbar \omega \hat{a}^{\dagger} \hat{a} + \hbar g \left( \hat{a}^{\dagger} \sigma_- + \hat{a} \sigma_+ \right) \]

describes a spin in an effective magnetic field, i.e.

\[ H = \hat{\mathbf{J}} \cdot \mathbf{B} \]

where the components of the vector \( \hat{\mathbf{J}} \) are given by Eqs (2).

3. Using the form (5) find the spectrum of the Hamiltonian (4)