Quantum Optics and Quantum Information SS 2006
Exercise IV (Wednesday 03.05.06)

I.  P AND WIGNER FUNCTIONS

1. Show that the Wigner function is a Gaussian convolution of the P function, i.e.
\[ W(\alpha) = \frac{2}{\pi} \int P(\beta) \exp \left[-2 |\beta - \alpha|^2\right] d\beta \]

2. Show that the Wigner function in suitable variables
\[ \alpha = \sqrt{\frac{m\omega}{2\hbar}} x + \frac{i}{\sqrt{2m\hbar}} p \]
can be represented as
\[ W(\alpha) = \frac{1}{\pi} \int \left( x + \frac{y}{2} \hat{\rho} |x - \frac{y}{2} \right \exp \left(-\frac{py}{\hbar}\right) dy \]

Hint: Express \( \hat{a} \) and \( \hat{a}^{\dagger} \) in terms of position and momentum operators.

3. Evaluate the Wigner function for the Fock state \( |n = 1\rangle \) and show that it can be negative.

4. Calculate
\[ \text{Tr} [\langle n | \exp (\eta \hat{a}^{\dagger} - \eta^{*} \hat{a}) | n \rangle] \]
characteristic function of a Fock state \( |n\rangle \).

II. GAUSS INTEGRALS

1. Show that
\[ \int_{\mathbb{R}^d} \exp \left(-\frac{1}{2} \langle x | A | x \rangle\right) dx = \left( \det \frac{A}{2\pi} \right)^{-1/2}, \]
where \( A = (A_{ij}) \) be a real \( d \times d \) positive-definite matrix, \( x = (x_1, \ldots, x_d) \) the Euclidean coordinates in \( V = \mathbb{R}^d \).

Hint: Diagonalize \( A \) by an orthogonal transformation (which does not change the integral) and apply a well-known formula for the Gauss integral
\[ \int_{-\infty}^{\infty} \exp \left(-\frac{1}{2} ax^2\right) dx = \sqrt{\frac{2\pi}{a}} \]

2. Obtain the more general form of the integral (1)
\[ \int_{\mathbb{R}^d} \exp \left(-\frac{1}{2} \langle x | A | x \rangle + \langle b | x \rangle\right) dx = Z_0 \exp \left(\frac{1}{2} \langle b | A^{-1} | b \rangle\right), \]
where
\[ Z_0 = \int_{\mathbb{R}^d} \exp \left(-\frac{1}{2} \langle x | A | x \rangle\right) dx. \]

Hint: Change \( x \to x - A^{-1} \cdot b \) of coordinates.

3. Show that the symmetrized second moments of Gaussian distribution
\[ \langle x_i x_j \rangle := \frac{1}{Z_0} \int_{\mathbb{R}^d} x_i x_j \exp \left(-\frac{1}{2} \langle x | A | x \rangle\right) dx \]
can be represented as \( (A^{-1})_{ij} \).