I. SQUEEZED STATES

A. Squeeze operator

1. By using the Baker-Hausdorff formula:

\[ e^{A}B e^{-A} = B + \frac{1}{1!} [A, B] + \frac{1}{2!} [A, [A, B]] + \frac{1}{3!} [A, [A, [A, B]]] + \ldots \]

show that

\[
\hat{S}(\varepsilon) \hat{a} \hat{S}^\dagger(\varepsilon) = \hat{a} \cosh r - \hat{a}^\dagger e^{2i\varphi} \sinh r, \\
\hat{S}(\varepsilon) \hat{a}^\dagger \hat{S}^\dagger(\varepsilon) = \hat{a}^\dagger \cosh r - \hat{a} e^{-2i\varphi} \sinh r,
\]

and

\[
\hat{S}(\varepsilon) f (\hat{a}, \hat{a}^\dagger) \hat{S}^\dagger(\varepsilon) = f \left( \hat{a} \cosh r - \hat{a}^\dagger e^{2i\varphi} \sinh r, \hat{a}^\dagger \cosh r - \hat{a} e^{-2i\varphi} \sinh r \right),
\]

where

\[
\hat{S}(\varepsilon) = \exp \left[ \frac{1}{2} \left( \varepsilon^* \hat{a}^2 - \varepsilon (\hat{a}^\dagger)^2 \right) \right], \quad \varepsilon = re^{2i\varphi}.
\]

2. The two possible orders in the \( D \) and \( S \) operators in the definition of a squeezed state in Eq. (36) (see the lecture) are not mathematically equivalent, since

\[
\hat{D}(\alpha) \hat{S}(\varepsilon) = \hat{S}(\varepsilon) \hat{D}(\gamma).
\]

Find \( \alpha \) in terms of \( \gamma \) and \( \varepsilon \).

Hint: Use the transformation (1).

B. Properties of Squeezed States

Calculate the mean photon number

\[ \bar{n} = \langle n \rangle = \langle \alpha, \varepsilon | \hat{n} | \alpha, \varepsilon \rangle = \langle \alpha, \varepsilon | \hat{a}^\dagger \hat{a} | \alpha, \varepsilon \rangle \]

and the variance of photon probability distribution

\[ (\Delta n)^2 = \langle \alpha, \varepsilon | \hat{n}^2 | \alpha, \varepsilon \rangle - \langle \bar{n} \rangle^2 \]

for the squeezed state \( |\alpha, \varepsilon\rangle \).

II. GENERALIZED PHOTON STATES

The usual coherent states are generated from the vacuum by the displacement operator

\[ D(\alpha) = G_1(\alpha) = \exp(\alpha^* \hat{a} - \alpha \hat{a}^\dagger), \]

whereas the squeezed states are generated from the vacuum by squeeze operator

\[ S(\alpha) = G_2(\alpha) = \exp(\alpha^* \hat{a}^2 - \alpha (\hat{a}^\dagger)^2). \]
It seems natural to suppose that one could define a much more general class of states, acting on the vacuum by operator

$$G_k (\alpha) = \exp \left( \alpha^* a^k - \alpha (a^\dagger)^k \right).$$  \hspace{1cm} (2)

Show that the vacuum expectation value $\langle 0 | G_k (\alpha) | 0 \rangle$ has zero radius of convergence as a power series with respect to $\alpha$, for $k > 2$.

Hint: Look at the matrix element

$$\langle 0 | G_k (\alpha) | 0 \rangle = 1 - |\alpha|^2 \frac{k!}{2!} + |\alpha|^4 \frac{1}{4!} \left((k!)^2 + 2k!\right) - |\alpha|^6 \frac{1}{6!} \left((k!)^3 + 2k!(2k)! + \frac{1}{k!} [2k!]^2 + (3k)! \right) +$$

$$+ ... + (-1)^n |\alpha|^{2n} \frac{1}{(2n)!} C_n (k) + ...$$

by expanding it as a Taylor series in $\alpha$ .

III. COHERENT PHASE STATES

The state of the classical oscillator can be described either in terms of its quadrature components $x$ and $p$, or in terms of the amplitude and phase, so that $x + ip = A \exp (i\varphi)$. In classical mechanics one can introduce the action and angle variables, which have the same Poisson brackets as the coordinate and momentum. However, in the quantum case we meet serious mathematical difficulties trying to define the phase operator in such a way that the commutation relation $[\hat{n}, \hat{\varphi}] = i$ would be fulfilled, where $\hat{n} = \hat{a}^\dagger \hat{a}$. One can introduce the exponential phase operators

$$\exp (i\hat{\varphi}) = \hat{E}_- = (\hat{a}^\dagger \hat{a} + 1)^{-1/2} \hat{a}$$

$$\exp (-i\hat{\varphi}) = \hat{E}_+ = \hat{a}^\dagger (\hat{a}^\dagger \hat{a} + 1)^{-1/2}$$

which can be considered, to a certain extent, as a quantum analogue of the classical phase $\exp (i\varphi)$.

1. Show that the normalizable state

$$|\varepsilon\rangle = \sqrt{1 - |\varepsilon|^2} \sum_{n=0}^{\infty} \varepsilon^n |n\rangle, \hspace{1cm} |\varepsilon| < 1 \hspace{1cm} (3)$$

is an eigenstate of the operator $\hat{E}_-$.

2. Show that the pure quantum state (3) has the same probability distribution $|\langle n |\varepsilon\rangle|^2$ as the mixed thermal state described by the density operator

$$\rho = \frac{1}{1 + \bar{n}} \sum_{n=0}^{\infty} \left( \frac{\bar{n}}{1 + \bar{n}} \right)^n |n\rangle \langle n|$$

if one identifies the mean photon number $\bar{n} \equiv \langle n \rangle$ with $|\varepsilon|^2 / \left(1 - |\varepsilon|^2 \right)$.

3. Show that for a coherent state $|\alpha\rangle$, where $\alpha = |\alpha| \exp (i\theta)$

$$\langle \alpha | \cos \hat{\varphi} | \alpha \rangle \approx \cos \theta$$

$$\langle \alpha | \sin \hat{\varphi} | \alpha \rangle \approx \sin \theta$$

for large $|\alpha| >> 1$. Thus the coherent state $|\alpha\rangle$ corresponds to a classical wave with phase $\theta$.

Hint:

$$\langle \alpha | \cos \hat{\varphi} | \alpha \rangle = |0\rangle D\dagger (\alpha) \left( \frac{\hat{a}^\dagger + \hat{a}^\dagger \hat{a} + 1}{2} \right) D (\alpha) |0\rangle =$$

$$= \langle 0 \rangle \left( (\hat{a}^\dagger + \alpha^\dagger) (\hat{a} + \alpha) + 1 \right)^{-1/2} \frac{1}{2} \left( (\hat{a}^\dagger + \alpha^\dagger) (\hat{a} + \alpha) + 1 \right)^{-1/2}$$

$$\approx \langle 0 \rangle \left( \frac{\bar{a} + \alpha}{2 |\alpha|} \right) |0\rangle \text{ for large } |\alpha| >> 1$$