Quantum Optics and Quantum Information SS 2006
Exercise II (Wednesday 19.04.06)

I. COHERENT STATES

1. Show that an eigenstate of $\hat{a}^\dagger$ cannot exist.
Hint: Expand an eigenstate of $\hat{a}^\dagger$ in the basis of Fock states $|n\rangle$

$$|\beta\rangle = \sum_{n=0}^{\infty} |n\rangle \langle n | \beta\rangle .$$

Then you will see that, the creation operator produces a new superposition of occupations $|n\rangle$ with different range of occupancy than $|\beta\rangle$.
2. Show

$$\hat{a}^\dagger |\alpha\rangle = \frac{\partial}{\partial \alpha} |\alpha\rangle ,$$

where $|\alpha\rangle$ is a coherent state, i.e. $\hat{a}|\alpha\rangle = \alpha |\alpha\rangle$.

II. "DISPLACEMENT" OPERATOR

By using $e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]}$,which holds if both A and B commute with $[A,B]$, show that

$$\hat{D}^\dagger(\alpha)\,\hat{a}\hat{D}(\alpha) = \hat{a} + \alpha,$$
$$\hat{D}^\dagger(\alpha)\,\hat{a}^\dagger\hat{D}(\alpha) = \hat{a}^\dagger + \alpha^*,$$
$$\hat{D}(\alpha+\beta) = \hat{D}(\alpha)\,\hat{D}(\beta)\,e^{-i\,\mathrm{Im}(\alpha\beta^*)},$$

where

$$D(\alpha) = \exp\left(\alpha\hat{a}^\dagger - \alpha^*\hat{a}\right).$$

III. PROPERTIES OF THE ELECTRIC FIELD OPERATOR

1. Show that, the mean electric field for an electric field operator

$$E = i\sqrt{\frac{2\pi\hbar\omega}{V}}\left[\hat{a}e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t} - \hat{a}^\dagger e^{-i\mathbf{k}\cdot\mathbf{x}+i\omega t}\right] \tag{1}$$

in a state with definite photon number is zero, i.e.

$$\langle n|\,E\,|n\rangle = 0.$$

2. Calculate the mean electric field in the coherent state $|\alpha\rangle$

$$\langle\alpha|\,E\,|\alpha\rangle ,$$

where $\alpha = |\alpha|\exp(i\theta)$. Show that it corresponds to a classical wave with phase θ.
3. What is the probability of obtaining n photons in the field (1)?
4. Solve the Schrödinger equation to obtain evolution of the coherent state $|\alpha\rangle$,

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle .$$

Hint: Solve the Schrödinger equation for Fock states

$$i\hbar\frac{\partial}{\partial t}|n\rangle = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)|n\rangle .$$