Quantum Information II (Exercises III)

1. **Witness operators**

(a) **Genuine N-party entanglement witness for cluster states**

First show that the witness operator \( W_{CN} = \frac{1}{2} \mathbb{1} - |\Phi\rangle_C \langle \Phi|_C \) detects the Cluster state \( |\Phi\rangle_C \), i.e. \( \langle \Phi|_C W_{CN} |\Phi\rangle_C < 0 \). Second, prove that this witness is non-negative on all separable states \( \rho_{\text{sep}} \), i.e. \( \text{tr}(W_{CN} \rho_{\text{sep}}) \geq 0 \).

**Hint:** It is sufficient to show that the witness is positive on all pure bi-separable states and convexity ensures positivity on all mixtures of these states. Show that the Schmidt coefficients of any bipartite representation of \( |\Phi\rangle_C \) cannot be so large such that the witness is negative on bi-separable states.

(b) **Optimal witness for the three qubit W state**

The witness which detects the state \( |W\rangle = \frac{1}{\sqrt{3}} (|001\rangle + |010\rangle + |100\rangle) \) is given by \( W_W = x \mathbb{1} - |W\rangle \langle W| \), with

\[
\text{tr}(W_W |W\rangle \langle W|) = x - |\langle W|W\rangle|^2 = x - 1 < 0 \quad \text{and} \quad \text{tr}(W_W \rho_{\text{sep}}) = x - \langle W| \rho_{\text{sep}} |W\rangle \geq 0.
\]

Find the optimal witness by calculating the maximal \( x \).

**Hint:** Use the hint of problem 1a.

(c) **Robustness of the witness \( W_W \) against white noise**

How much white noise \( p \) can maximally be added to the state \( |W\rangle \), i.e. \( \rho_W = \frac{p}{8} \mathbb{1} + (1 - p) |W\rangle \langle W| \), to still detect \( \rho_W \) with the witness \( W_W \) of exercise 1b?
2. Graph states

(a) Measurement on a five qubit ring graph

Consider the five qubit ring graph state.

\[
\begin{array}{c}
1 \\
2 \\
3 \\
4 \\
5
\end{array}
\]

Draw the remaining graphs \((G')\) after the following Pauli measurements on qubit 1:

i. \(\sigma_z\), which gives \(G' = G \setminus \{a\}\),

ii. \(\sigma_y\), which gives \(G' = G \setminus E(N_a, N_a)\) and

iii. \(\sigma_x\), which gives

\[
G' = G \triangle E(N_{b_0}, N_a) \triangle E(N_{b_0} \cap N_a, N_{b_0} \cap N_a) \triangle E(\{b_0\}, N_a \setminus \{b_0\}),
\]

using the definitions from the lecture.

(b) Spider graph state

Show that the spider graph state \(|G\rangle\),

\[
\begin{array}{c}
1 \\
2 \\
3 \\
4 \\
5 \\
6
\end{array}
\]

corresponds (up to local transformations) to a six qubit GHZ state

\[
(|GHZ_j\rangle = \frac{1}{\sqrt{2}} (|j\rangle |0\rangle \pm |11111 - j\rangle |1\rangle), \text{ with } j = 00000, \ldots, 11111 \). \]

Hint: Show that an "adjusted" GHZ state, i.e. a GHZ state plus suitable local unitary transformations is an eigenstate of the stabilizer \((S_G = \{K^{(a)}_G\}_{a \in V}\)). You can alternatively generate the graph state \(|G\rangle\) by \(|G\rangle = \Pi_{(a,b) \in E} S^{(a,b)} |+\rangle^\otimes 6\) and show that this state can be transformed into a GHZ state by local unitary transformations.