Quantum Information II (Exercises II)

1. Cluster States

(a) Maximal connectedness of Cluster states (Lemma (10 i))

Show that the linear cluster state of $N > 3$ particles, $|\phi_N\rangle$, is maximally connected. Use the fact that $|\phi_N\rangle$ can be written as

$$|\phi_N\rangle = \frac{1}{2^{N/2}} \bigotimes_{a=1}^{N} (|0\rangle_a \sigma_z^{(a+1)} + |1\rangle_a), \quad (14)$$

with the convention $\sigma_z^{N+1} \equiv 1$ and $\sigma_z^{(a)}$ meaning $\sigma_z$, acting on qubit $a$. Note that this is a recursion formula, $|\phi_{N+1}\rangle = 1/\sqrt{2}(|0\rangle_0 \sigma_z^{(1)} + |1\rangle_0) |\phi_N\rangle$, where we have added a particle at the beginning of the chain. Proceed in the following way:

i. Show that a measurement of $\sigma_x$ at the second qubit of the $N$-particle state results in the state

$$|\phi_N\rangle \xrightarrow{\sigma_x^{(2)}} \frac{1}{\sqrt{2}} (1 - i\sigma_y^{(1)}) |\phi_{N-1}\rangle, \quad (15)$$

if the measurement outcome is 0, and

$$|\phi_N\rangle \xrightarrow{\sigma_x^{(2)}} \frac{1}{\sqrt{2}} (\sigma_x^{(1)} + \sigma_z^{(1)}) |\phi_{N-1}\rangle, \quad (16)$$

if the measurement outcome is 1, where $|\phi_{N-1}\rangle$ is a linear cluster state consisting of particles $1, 3, \ldots, N$.

ii. Use the previous result to show that measuring $\sigma_x$ on each intermediate particle $2, \ldots, N-1$ leads to a two-particle linear cluster state, up to a unitary transformation on the first particle,

$$|\phi_N\rangle \xrightarrow{\sigma_x^{(2)} \otimes \cdots \otimes \sigma_x^{(N-1)}} U |\phi_2\rangle, \quad (17)$$

where $|\phi_2\rangle$ is a linear cluster state consisting of particles 1 and $N$. Calculate also the possible expressions of $U$.

This proves that the two “outer” particles can be brought into a maximally entangled state by measuring $\sigma_x$ on the “inner” particles.

iii. Conclude the proof by showing how to remove the particles $1, \ldots, j - 1$ and $k + 1, \ldots, N$, if one has to create a maximally entangled state between particle $j > 1$ and $k < N$ with $j < k$. (Hint: Remember the lecture about the one-way quantum computer.)
(b) **Persistence of a Cluster state (Lemma (10 ii))**

Prove that the persistency $P_e$ of a linear cluster state of $N$ particles, $|\phi_N\rangle$, is equal to $\lfloor N/2 \rfloor$.

i. Show that $P_e(|\phi_N\rangle) \leq \lfloor N/2 \rfloor$.

ii. Show that $P_e(|\phi_N\rangle) \geq \lfloor N/2 \rfloor$. Use the fact that the Schmidt measure of $|\phi_N\rangle$, i.e. the logarithm of the minimal number of terms in a decomposition of $|\phi_N\rangle$ in a product basis, is equal to $\lfloor N/2 \rfloor$. Note that since $P_e$ measurements disentangle $|\phi_N\rangle$, there exists a decomposition

$$|\phi_N\rangle = \sum_{j_1,\ldots,j_{P_e}=0}^{1} \mu^{(j_1)}_{1} \otimes \cdots \otimes \mu^{(j_{P_e})}_{P_e} \otimes |\psi_{j_1,\ldots,j_{P_e}}\rangle,$$

where the $|\mu^{(j_k)}_k\rangle$ are the 1-qubit eigenstates of observable $k$, corresponding to measurement outcome $j_k$. $a_k$ is the number of the measured qubit, $1 \leq k \leq P_e$, and $|\psi_{j_1,\ldots,j_{P_e}}\rangle$ is some unnormalized *product* state of the remaining qubits. (Why?)

(c) **State transport via spin chains**

A basic ingredient to build a one-way quantum computer is the ability to implement “quantum wires”, i.e. using a linear cluster of $N+1$ states (spin chain) to transport a state $|\psi\rangle$ from one end to the other end via local measurements.

i. Generate a linear $(N+1)$-qubit cluster with input state $|\psi\rangle$ from the initial state $|\psi\rangle \otimes |+\rangle \otimes |\psi\rangle$.

ii. Construct a deterministic set of local measurements and eventually local rotations such that after the procedure the cluster state is of the form $(\otimes_{i=1}^{N} |s_i\rangle) \otimes |\psi\rangle$.

iii. Discuss the similarities and differences between an analogous transport of a quantum state via quantum teleportation and the transport by the "quantum wire".